May 2026 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 5 questions with 4 parts. Each part is worth 5 points for a total of 100 points. There are four pages including this one.

In what follows all rings R are assumed to have a multiplicative identity 1_R. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example.

1. Groups.

(a) Prove that any two bases of a finitely generated free abelian group have the same cardinality.

(b) Prove that if P is a Sylow p-subgroup of a finite group G then $N_G(N_G(P)) = N_G(P)$.

(c) Let $\mathbb{F}_q$ be a finite field of order $q = p^n$ with p prime. Let $B \subset \mathrm{GL}_n(\mathbb{F}_q)$ be the subgroup of $n \times n$ invertible upper triangular matrices, and let $U \subset B$ be the subgroup of upper triangular matrices with ones on the diagonal.
   (i) What is the order B?
   (ii) Prove U is the unique p-Sylow subgroup of B.

(d) With the notation as in part (c) prove that B is solvable.

2. Rings.

(a) Prove that if R is a commutative ring and $\mathfrak{p}$ is a prime ideal then $R_{\mathfrak{p}}$ is a local ring.

(b) Let R be a commutative ring.
   (i) Prove that if $s \in R$ is nilpotent and $S = \{s^n | n \geq 0\}$ then $S^{-1}R = 0$.
   (ii) Prove that the set of nilpotent elements in a commutative ring R is the intersection of all prime ideals.

(c) Find a pair of orthogonal idempotents in the ring $R = \mathbb{Q}[x]/(x^3 - 1)$, and use them to express R as a product of two fields.

(d) Let R be a UFD. Prove that if f, g are primitive polynomials in $R[x]$ then fg is also primitive.

3. Fields.

(a) Prove that if $f \in \mathbb{Q}[x]$ is an irreducible cubic polynomial with exactly one real root then the Galois group of f is the group S_3.

(b) (i) Give an example of an extension $F \supset E \supset K$ where F is Galois over E, and E is Galois over K but F is not Galois over K.

(ii) Prove or give a counter-example. If $F \supset K$ is a field extension of finite degree then there are at most finitely many intermediate fields $F \supset E \supset K$.

(c) Prove that any finite subgroup of the multiplicative group of units in a field is cyclic.

(d) Let $x_1, \ldots, x_n$ be indeterminates and let $e_1, \ldots, e_n$ be the elementary symmetric polynomials in $x_1, \ldots, x_n$. Let $F = \mathbb{Q}(e_1, \ldots, e_n)$ and let $f = t^n - e_1 t^{n-1} + e_2 t^{n-2} + \ldots + (-1)^n e_n \in F[t]$. Prove that the Galois group of f is the symmetric group S_n. (Hint: The elementary symmetric polynomials are invariant under permutation.)

4. Modules.

(a) Let $R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x+1))$. Prove that the ideal (x, y) is a projective R-module.

(b) Let R be a commutative ring and let S be a multiplicatively closed subset. If M is an R-module then we define the $S^{-1}R$ module $S^{-1}M = \{\frac{m}{s} | m \in M, s \in S\}$ where $\frac{m}{s} = \frac{m'}{s'}$ in $S^{-1}M$ if $s(s'm - sm') = 0$ for some $s \in S$.

(i) What is the kernel of the map of R-modules $M \to S^{-1}M$, $m \mapsto \frac{m}{1}$?

(ii) Prove that if $\phi\colon M \to N$ is an monomorphism of R-modules then the induced map of $S^{-1}R$-modules $S^{-1}M \to S^{-1}N$ is injective.

(c) Prove that if R is a non-zero commutative ring such that every submodule of a free module is free then R is a principal ideal domain.

(d) Let R be a domain and let A be a non-zero R-module. Prove or give a counter example to the following statement. *The dual module* $A^* = \mathrm{Hom}(A, R)$ *is non-zero.*

5. Linear Algebra.

(a) Give an explicit example of a non-zero alternating bilinear form $(\mathbb{R}^3)^2 \to \mathbb{R}$.

(b) Let $T\colon V \to V$ be a linear transformation of an n-dimensional vector space.
   (i) Prove that if V is T-cyclic then $\operatorname{rank} T \geq n-1$.
   (ii) Does the converse hold? Ie, if $\operatorname{rank} T \geq n-1$ is V T-cyclic? Prove or give a counter example.

(c) Let G be a finite group. Prove that if V is a finitely generated simpl $\mathbb{C}[G]$-module then any endomorphism of $\mathbb{C}[G]$-modules $V \to V$ is a scalar multiple of the identity. (This result is known as Schur's lemma).

(d) Find the possible Jordan canonical forms for a matrix whose characteristic polynomial is $(x-1)(x^3-1)$.