# Qualifying Examination in Analysis

**May 2026**

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Solve all problems.

1. (a) Give a reasonable description/characterization of the Borel-Stieltjes measure on $\mathbb{R}$ associated to a function $F : \mathbb{R} \to \mathbb{R}$, under suitable conditions on $F$.

   (b) Let $F : \mathbb{R} \to \mathbb{R}$ be defined as

   $$F(x) = \begin{cases} -2 & \text{for } x < 1 \\ \ln x & \text{for } x \geq 1. \end{cases}$$

   and let $\mu_F$ be the associated Borel-Stieltjes measure on $\mathbb{R}$. Prove that the (essentially unique) Radon-Nikodym decomposition of $\mu_F$ with respect to the Lebesgue measure $m$, both viewed as measures on the Borel $\sigma-$algebra $\mathcal{B}_{\mathbb{R}}$, is given as

   $$\mu_F = 2\delta_1 + \rho, \qquad \text{where} \quad d\rho = f dm, \qquad f(x) = \frac{1}{x}\,\chi_{(1,\infty)}(x),$$

   and where $\delta_1$ denotes the Dirac delta measure at 1.
   [Note: proving that $\mu_F = 2\delta_1 + \rho$, with $\rho$ as above, is only part of the answer.]

2. Let $X$ be any nonempty set and $A \subseteq X$, a fixed nonempty subset, and let

   $$\mathcal{F}_A = \{E \subseteq X : A \subseteq E \ \text{ or } \ E \subseteq A^c\}.$$

   It turns out that $\mathcal{F}_A$ is a $\sigma$-algebra (you do not need to prove this).

   (a) Prove that $f : X \to \mathbb{R}$ is $(\mathcal{F}_A, \mathcal{B}_{\mathbb{R}})$-measurable if and only if $f$ is constant on $A$.

   (b) If $X = \mathbb{R}$ and $A = \mathbb{R} \setminus \mathbb{Q}$, then prove that $\mathcal{F}_A \subseteq \mathcal{L}$, the class of Lebesgue measurable sets on $\mathbb{R}$, and that the inclusion is proper.

   (c) If $m$ denotes the Lebesgue measure, and $X, A$ are as in b), then prove that $L^1(X, \mathcal{F}_A, m_{|\mathcal{F}_A})$ (intended in the sense of equivalence classes of $(\mathcal{F}_A, \mathcal{B}_{\mathbb{R}})$-measurable functions) consists only of the zero element.

3. (a) State Fatou's Lemma.
   (b) Given the sequence $f_n : [0,1] \to \mathbb{R}$ defined as $f_n(x) = 3 + (-1)^n x + n^a x^n$, find whether or not equality holds in the conclusion of Fatou's Lemma, for each of the following: $a = 1/2, a = 1, a = 2$.

4. (a) Define convergence in measure on a measure space $(X, \mathcal{M}, \mu)$.
   (b) Given the sequence on $(\mathbb{R}, \mathcal{L}, m)$

$$f_n(x) = \begin{cases} nx^{-\frac{1}{2}} & \text{if } 0 < x \le \frac{1}{n+1} \\ e^{-nx} & \text{if } x > \frac{1}{n+1} \\ 0 & \text{if } x \le 0 \end{cases} \tag{1}$$

   establish that the sequence $\{f_n\}$ converges pointwise and in measure, but not in $L^1$.

5. (a) State Tonelli's part of Fubini-Tonelli's theorem (for nonnegative functions).
   (b) Let

$$E = \{(x,y) \in \mathbb{R}^2 : \ x \in [0,2], \ 0 \le y \le x\},$$

   and let $\mu_F$ be the measure in problem 1. Compute $\mu_F \otimes m(E)$.

6. (a) Define bounded linear operator between two normed spaces, and its operator norm.
   (b) Let $c_{00}$ be the vector space of real-valued sequences with finite support (i.e. which are non zero for at most finitely many indices), equipped with the $\ell^\infty$ norm. Prove that the linear operator $S : c_{00} \to c_{00}$ defined as $S(\{x_n\}) = \{nx_n\}$ is unbounded.
   (c) Prove that the linear operator

$$Tf(x) = \int_x^1 tf(t)dt$$

   is well defined and bounded from $L^3([0,1])$ to $L^1([0,1])$ (with respect to the Lebesgue measure).
   (d) Compute $\|T\|$.

7. Let $H$ be a Hilbert space over $\mathbb{C}$ and let $E, F$ be closed subspaces of $H$. Let $\alpha \in [0,1)$ and suppose that

$$|\langle x, y\rangle| \le \alpha \|x\| \, \|y\|, \qquad x \in E, \ y \in F.$$

   (a) Prove that $\|x + y\|^2 \ge (1 - \alpha)(\|x\|^2 + \|y\|^2)$ for $x \in E$ and $y \in F$.
   (b) Show that the set $E + F = \{x + y : x \in E, y \in F\}$ is closed and $E \cap F = \{0\}$.
   (c) Suppose that $u \in E$ and $v \in F$ are unit vectors. Prove that there exists a unique vector $w \in E + F$ with $\|w\| \le \sqrt{\frac{2}{1-\alpha}}$ such that

$$\langle x + y, w\rangle = \langle x, u\rangle + \langle y, v\rangle \quad x \in E, \ y \in F.$$