Analysis Qualifying Exam –August 2025

Instructions: *Do all 8 problems. Use a separate sheet for each problem. Your work will be graded for correctness, completeness, and clarity.*

1. Let $\mathcal{E} = \{\varnothing, \mathbb{R}\} \cup \{(0, a) : a > 0\}$, and let $\sigma(\mathcal{E})$ be the $\sigma-$algebra (on $\mathbb{R}$) generated by $\mathcal{E}$.

   a. Prove, in this order, that $\sigma(\mathcal{E})$ contains the following intervals:

   $$\text{(i) } [a, b),\ a > 0; \quad \text{(ii) } (a, b),\ a \geqslant 0; \quad \text{(iii) } (a, \infty),\ a \geqslant 0; \quad \text{(iv) } (-\infty, 0].$$

   b. If $\mathcal{B}_{(0,\infty)}$ denotes the Borel σ-algebra on $(0, \infty)$, prove that $\mathcal{B}_{(0,\infty)} \subseteq \sigma(\mathcal{E})$.

   c. Prove that

   $$\sigma(\mathcal{E}) = \big\{A \cup B :\ A \in \{\varnothing, (-\infty, 0]\,\},\ B \in \mathcal{B}_{(0,\infty)}\big\}. \tag{1}$$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space.

   a. Give the definition of convergence in measure for a sequence $\{f_n\}$ on the space $(X, \mathcal{A}, \mu)$.

   b. Assume $\mu(X) < \infty$ and define, for $f : X \to \mathbb{R}$ measurable

   $$\rho(f) = \int_X \frac{|f|}{1 + |f|} d\mu.$$

   Prove that the function $(f, g) \mapsto \rho(f - g)$ is a metric on the space of measurable functions on X (identifying functions that are equal a.e.). [Hint: $x \to \frac{x}{1+x}$ is increasing on $[0, \infty)$.]

   c. If $\mu(X) < \infty$, prove that $\rho(f_n) \to 0$ if and only if $f_n \to 0$ in measure.

3. Use Lebesgue's dominated convergence theorem to prove that

   $$\lim_{n\to\infty} \int_0^n \frac{1}{\sqrt{x}} \left(\frac{\sin x}{x}\right)^n dx = 0.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^{-1/2}\mathbb{1}_{(0,1)}(x)$. Let $\{r_n\}$ be an enumeration of the rationals, and let

   $$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n), \qquad x \in \mathbb{R}.$$

   a. Let $p \geqslant 1$. Prove that $g \in L^p(\mathbb{R})$ if and only if $p \in [1, 2)$.

   b. Prove that g is finite a.e., but it is unbounded on any interval $(a, b) \subseteq \mathbb{R}$.

5. Let $\{f_n\}$ be a sequence of measurable functions on a probability space $(X, \mathcal{A}, \mu)$. Suppose that

$$\sup_n \int_{[0,\infty)} t^{2/3} \mu(\{x \in X : |f_n(x)| > t\}) dt < \infty.$$

Prove that $\{f_n\}$ is uniformly integrable.

6. Let X be a vector space over $\mathbb{R}$ equipped with two non-equivalent norms.

   a. Show that there is a sequence $\{x_n\}$ in X that converges to 0 in one of the norms and is unbounded in the other.

   b. Prove that there exists a linear functional on X that is bounded with respect to one of the norms and unbounded with respect to the other.

7. Let $p \in (1, \infty)$ and let T be a linear map from $L^p[0,1]$ to itself with the property that if $\{f_n\}$ is a sequence in $L^p[0,1]$ that converges almost everywhere to some f in $L^p[0,1]$, then $\{Tf_n\}$ converges almost everywhere to Tf. Prove that T is bounded.

8. Let $p \in (2, \infty)$. For $f \in L^p[0,1]$, let $\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}$.

   a. Let $a, b \in \mathbb{R}$. Prove that

   $$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leqslant \frac{1}{2}(|a|^p + |b|^p).$$

   b. Let $f, g \in L^p[0,1]$. Prove that

   $$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leqslant \frac{1}{2}(\|f\|_p^p + \|g\|_p^p).$$

   c. Prove that if C is a non-empty closed convex subset of $L^p[0,1]$, then there is a unique $f^* \in C$ such that

   $$\|f^*\|_p = \inf_{f \in C} \|f\|_p.$$