Qualifying Examination in Analysis

August 2024

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Solve all problems.

1. Suppose that $E \subset \mathbb{R}$ is a Lebesgue measurable set with $m(E) < \infty$. Consider the function $f(x) = m(E \cap (-\infty, x])$, $x \in \mathbb{R}$. Show the following:
   (a) f is Lipschitz on $\mathbb{R}$, that is, there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$, for all $x, y \in \mathbb{R}$.
   (b) $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = m(E)$.
   (c) $f(\mathbb{R}) \supseteq (0, m(E))$.
   (d) If $0 < \alpha < m(E)$, then there exists a compact set K such that $K \subset E$ and $m(K) = \alpha$.
   (e) With an example show that the conclusion in (e) may not hold, if $\alpha = m(E)$.

2. Consider the following spaces of sequences:

$$\ell_c^\infty = \{x : \mathbb{N} \to \mathbb{C} \text{ such that } x \text{ is eventually constant } \},$$

$$c = \{x : \mathbb{N} \to \mathbb{C} \text{ such that } \lim_{n \to \infty} x(n) \text{ exists } \}.$$

   (a) Prove that c is a subspace of ℓ^∞ and a Banach space.
   (b) Prove that ℓ_c^∞ is a subspace of c, but it is not a Banach space.

3. For each $t \in \mathbb{R}$ let

$$F(t) = \int_{\{x \in \mathbb{R}^3 \colon |x| \geq 1\}} \frac{|x|^t}{1 + |x|^4} dx,$$

   where the integral is taken with respect to the Lebesgue measure on $\mathbb{R}^3$.
   (a) Show that $F(t) < \infty$ for $-\infty < t < 1$.
   (b) Using explicitly Lebesgue's Dominated Convergence Theorem, show that $\lim_{t \to -\infty} F(t) = 0$.
   (c) Using explicitly Fatou's Lemma, show that $\lim_{t \to 1^-} F(t) = +\infty$.

4. Let X be the vector space of real-valued polynomials on $\mathbb{R}$, and for $p \in X$ let

$$\|p\| = \sup_{t\in[-1,1]} |p(t)|.$$

   (a) Show that $\|\cdot\|$ defined above is indeed a norm on X (you may use the Fundamental Theorem of Algebra).

   (b) Show that the operator $T : X \to X$ defined as $T(p) = p'$ is linear but not bounded.

   (c) For any $a, b \in \mathbb{R}$ consider the functional $f : X \to \mathbb{R}$ defined as $f(p) = p(a) + p(b)$. Show that f is linear, and that it is bounded if and only if $a, b \in [-1, 1]$.

5. Let X be a nonempty set, and let $Y \subseteq X$. For any family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ let $\mathcal{F}_Y = \{A \cap Y : \; A \in \mathcal{F}\}$.

   Given $\mathcal{E} \subseteq \mathcal{P}(X)$ and $Y \in \mathcal{E}$, let $\mathcal{A} = \{A \subseteq X : \; A \cap Y \in \sigma(\mathcal{E}_Y)\}$.

   (a) Prove that $\mathcal{A}$ is a $\sigma-$algebra on X.

   (b) Prove that $\sigma(\mathcal{E}_Y) = \big(\sigma(\mathcal{E})\big)_Y$.

   (c) Show that if (X, τ) is a topological space and $Y \subseteq X$ is a Borel set in X, then for any Borel set B in X, the set $B \cap Y$ is a Borel set in Y, endowed with the topology induced by τ.

   (d) Show that it is not true that intersecting a Lebesgue measurable set $E \subseteq \mathbb{R}^2$ with a Borel subset $Y \subseteq \mathbb{R}^2$ always gives a Lebesgue measurable set in Y.

6. Let m^k be the Lebesgue measure on $\mathbb{R}^k$, and let $\mathcal{B}_{\mathbb{R}^k}$ be the Borel σ-algebra on $\mathbb{R}^k$. Let $Y = \mathbb{R} \times \{0\}$ and $D = \{(x, y) : x^2 + y^2 \leq 1\}$. According to (c) of Problem 5, if $B \in \mathcal{B}_{\mathbb{R}^2}$ then $B \cap Y$ is a Borel set in Y. Hence, given any $B \in \mathcal{B}_{\mathbb{R}^2}$ there is a unique $B_1 \in \mathcal{B}_{\mathbb{R}}$ such that $B \cap Y = B_1 \times \{0\}$, and the set functions on $\mathcal{B}_{\mathbb{R}^2}$ defined as

$$m^2_Y(B) = m^1(B_1), \qquad m^2_D(B) = m^2(B \cap D),$$

   define Borel measures on $\mathbb{R}^2$ (you do not need to prove any of these assertions).

   Consider the signed measure on $\mathcal{B}_{\mathbb{R}^2}$ defined as

$$\nu = m^2_D - m^2_Y + \delta_{(2,0)},$$

   where $\delta_{(2,0)}$ is the Dirac measure at the point $(2, 0) \in \mathbb{R}^2$.

   (a) Find a Hahn decomposition for ν.

   (b) Find the Lebesgue decomposition of ν with respect to the Lebesgue measure m^2, as defined on $\mathcal{B}_{\mathbb{R}^2}$.

   Note: your answers in both (a) and (b) must be thoroughly justified.