Algebra Qualifying Exam
August 20, 2025

This exam has 8 questions for a total of 64 points. In order to receive full credit, SHOW ALL YOUR WORK.

You may use any result discussed in the Algebra I (Fall 24) and Algebra II (Spring 25) courses as part of your solution, without reworking the proof. However, the result you cite must not be essentially equivalent to the problem itself. Hints are suggestions only, and you are not required to use them.

- All rings are assumed to be associative and contain an identity element $1 \neq 0$.
- $\mathbb{Q}$ denotes the field of rational numbers.
- $\mathbb{N}$ denotes the set of all non-negative integers (including 0).
- Homomorphisms of R-modules are also referred to as R-linear maps.

Part I: Groups, Rings, and Fields

1. (6 points) Show that there are no simple groups of order 224.

2. Let G be a finite group whose only automorphism is the identity map, *i.e.*, if $f : G \to G$ is automorphism then $f(x) = x$ for all $x \in G$.

   (a) (4 points) Show that G is ableian and $x^2 = e$ for all $x \in G$, where e is the indentity element of G.

   (b) (8 points) Show that G is either the trivial group or $\mathbb{Z}_2$, the cyclic group of order 2.

3. Let R be a commutative ring with identity. An ideal I of R is called *irreducible* if whenever

   $$I = I_1 \cap I_2$$

   for some ideals I_1, I_2 of R, it follows that either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$.

   (a) (2 points) If P is a prime ideal of R, prove that P is irreducible.

   (b) (3 points) Let $x \in R$ be a nonzero element. Show that there exists an ideal I_x that is maximal among all ideals I of R with $x \notin I$. Furthermore, prove that I_x is irreducible.

   (c) (2 points) Assume that every irreducible ideal of R is a prime ideal. Prove that there are no non-zero nilpotent elements in R.

4. Let E be the splitting field over $\mathbb{Q}$ of $f(x) = (x^2 + 5)(x^2 - 7) \in \mathbb{Q}[x]$.

   (a) (5 points) Find the Galois group $\mathsf{Gal}(E/\mathbb{Q})$.

   (b) (2 points) Determine the number of intermediate fields $\mathbb{Q} \subsetneq K \subseteq E$ such that $K/\mathbb{Q}$ is a Galois extension.

Part II: Module Theory and Linear Algebra

1. Prove or give a counterexample. If your answer includes a counterexample, you must explain why it works in order to receive full credit.
   (a) (2 points) Let R be an integral domain and I an ideal of R. If R/I is a flat R-module then $I = \{0\}$.
   (b) (2 points) Let $n \geq 2$ be an integer. Then $\mathbb{Z}/n\mathbb{Z}$ is a projective $\mathbb{Z}$-module.
   (c) (4 points) Let V be a finite dimensional vector space over a field K. Let $T : V \to V$ be a linear transformation and $W \leq V$ a subspace of V that is T-invariant, i.e. $T(W) \subseteq W$. Let $C(x)$, $C_1(x)$, and $C_2(x)$ be the characteristic polynomial of T as a linear operator on V, W, and V/W, respectively. Then $C(x) = C_1(x) \cdot C_2(x)$.

2. Let R be a PID, $p \in R$ a prime element, and $F = R/(p)$.
   (a) (4 points) Let $M = R/(a)$. Then

$$M/pM \simeq \begin{cases} F & \text{if } p \text{ divides } a \\ \{0\} & \text{if } (a,p) = 1 \end{cases}$$

   (b) (2 points) Let $M = R/(a_1) \oplus \ldots \oplus R/(a_s)$ where p divides each a_i, $i \in \{1, \ldots, s\}$. Show that $M/pM \simeq F^s$.

3. (8 points) Let M_1 and M_2 be two R-modules. Show that $M_1 \oplus M_2$ is injective if and only if M_1 and M_2 are injective.

4. (a) (6 points) Let N_1 and N_2 be two 5×5 nilpotent matrices over a field F. Show that if N_1 and N_2 have the same rank and the same minimal polynomial then N_1 and N_2 are similar.
   (b) (4 points) Let A and B be two $n \times n$ matrices over a field F such that A and B have the same characteristic polynomial

$$c(X) = (X - \lambda_1)^{d_1} \cdot \ldots \cdot (X - \lambda_l)^{d_l},$$

   with $\lambda_1, \ldots, \lambda_l \in K$ pairwise distinct. Suppose furthermore that A and B have the same minimal polynomial, and that the matrices $A - \lambda_i \cdot I_n$ and $B - \lambda_i \cdot I_n$ have the same rank for all $i \in \{1, \ldots, l\}$, . If $d_i \leq 5$ for all $i \in \{1, \ldots, l\}$ show that A and B are similar.