Analysis Qualifying Exam – May 2025

Instructions: *Do all 8 problems. Use a separate sheet for each problem. Your work will be graded for correctness, completeness, and clarity.*

1. Let X be a non-empty set.
   a. Define the notions of a **semi-ring** $\mathcal{S}$ of subsets of X, and a **pre-measure** μ on $\mathcal{S}$.
   b. State Carathéodory's Theorem.
   c. Outline the main steps in the proof of Caratheodory's theorem; be sure to define **outer meausure** μ^* and **μ^*-measurability**.

2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $[0,1]$ such that $f_k \to f$ a.e. Prove that
$$\lim_{k\to\infty} \int_0^1 f_k(x)e^{-f_k(x)}dx = \int_0^1 f(x)e^{-f(x)}dx.$$

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leqslant p < q < \infty$.
   a. Prove that if $\mu(X) < \infty$, then $L^q(\mu) \subseteq L^p(\mu)$.
   b. Assume that $\sup\{\mu(A) : A \in \mathcal{A}, \mu(A) < \infty\} = \infty$.
      i. Show that there exist pairwise disjoint sets $A_i \in \mathcal{A}$ of finite measure such that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$.
      ii. Prove that there exists a function f in $L^q(\mu)\backslash L^p(\mu)$.

4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Assume that $f, g : X \to \mathbb{R}$ are measurable functions. For $t \in \mathbb{R}$, let
$$F_t = \{x \in X : f(x) > t\} \quad \text{and } G_t = \{x \in X : g(x) > t\}.$$
   a. Prove that
$$\int_X |f(x) - g(x)| d\mu(x) = \int_{\mathbb{R}} \mu((F_t\backslash G_t) \cup (G_t\backslash F_t))dt.$$
   b. Assume that f and g are also integrable. Explain why the identity in part (a) remains valid even when $(X, \mathcal{A}, \mu)$ is not assumed to be σ-finite.

5. Let $f : [0,1] \to [0,1]$ be absolutely continuous and let $g : [0,1] \to [0,1]$ be given by $g(x) = \sqrt{x}$.
   a. Prove that $f \circ g$ is absolutely continuous.
   b. Give an example to show that $g \circ f$ need not be absolutely continuous.

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{1+x^2}$.
   a. Show that the Fourier transform of f is given by $\widehat{f}(\xi) = \frac{1}{2}e^{-|\xi|}$.
   b. Prove that the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$H(x, y) = \frac{e^{-|x-y|}}{1 + (x + y)^2}$$
   is the Fourier transform of a function $G \in L^1(\mathbb{R}^2)$ (hint: use a change of variables).

7. Let E and F be Banach spaces and let $T : E \to F$ be a bounded linear operator satisfying $\|Tx\| > \|x\|$ for all $x \in E \backslash \{0\}$. Prove that the adjoint T^* of T is an open mapping.

8. Let H be a separable Hilbert space over $\mathbb{R}$ with orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in H is said to be *weakly Cauchy* if for each $y \in H$, $\{\langle x_n, y \rangle\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

   a. Prove that if $\{x_n\}$ is weakly Cauchy, then it is bounded.

   b. Show that if $\{x_n\}$ is weakly Cauchy, then there exists $x \in H$ such that $\langle x_n, y \rangle \to \langle x, y \rangle$ for each $y \in H$.

   c. Show that if $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$ and $\{\langle x_n, e_m \rangle\}_n$ converges for each m, then $\{x_n\}$ is weakly Cauchy.