Algebra Qualifying Exam
May 30, 2025

This exam has 8 questions for a total of 64 points. In order to receive full credit, SHOW ALL YOUR WORK.

You may use any result discussed in the Algebra I (Fall 24) and Algebra II (Spring 25) courses as part of your solution, without reworking the proof. However, the result you cite must not be essentially equivalent to the problem itself. Hints are suggestions only, and you are not required to use them.

- All rings are assumed to be associative and contain an identity element $1 \neq 0$.
- $\mathbb{Q}$ denotes the field of rational numbers.
- $\mathbb{N}$ denotes the set of all non-negative integers (including 0).
- Homomorphisms of R-modules are also referred to as R-linear maps.

Part I: Groups, Rings, and Fields

1. (5 points) Show that there are no simple groups of order 96.

2. (a) (2 points) Let G be a finitely generated group. Recall that a subgroup M of G is called a *maximal subgroup* if $M \neq G$ and the only subgroups of G which contain M are M and G.

   Prove that if H is a proper subgroup of G (*i.e.* $H \subsetneq G$) then there exists a maximal subgroup of G containing H.

   (b) (4 points) Let G be a finite group and H a subgroup of G such that

$$\bigcup_{g \in G} gHg^{-1} = G.$$

   Prove that $H = G$.

   (c) (6 points) Let G be a non-trivial finite group such that any two maximal subgroups are conjugate, *i.e.*, if M and M' are two maximal subgroups of G then $M' = gMg^{-1}$ for some $g \in G$. Prove that G is a cyclic group of a prime power order.

   [*Hint: Use parts* (*a*) *and* (*b*) *to show that* G *is cyclic.*]

3. Let R be a commutative ring. For any ideal I of R, define

$$V(I) := \{P \mid P \text{ is a prime ideal of } R \text{ with } I \subseteq P\}.$$

   (a) (3 points) Let $e \in R$ be an idempotent element (*i.e.*, $e^2 = e$). Show that

$$V(R(1-e)) = \{P \mid P \text{ is prime with } Re \not\subseteq P\}.$$

   (b) (5 points) Let e_1 and e_2 be two idempotent elements of R. Show that if

$$V(Re_1) = V(Re_2)$$

   then

$$e_1 = e_2.$$

   [*Hint: Think about the intersection of all prime ideals containing a particular ideal.*]

4. Let E be the splitting field over $\mathbb{Q}$ of $f(x) = (x^2+3)(x^2-5) \in \mathbb{Q}[x]$.

   (a) (5 points) Find the Galois group $\mathsf{Gal}(E/\mathbb{Q})$.

   (b) (2 points) Determine the number of intermediate fields $\mathbb{Q} \subsetneq K \subseteq E$ such that $K/\mathbb{Q}$ is a Galois extension.

Part II: Module Theory and Linear Algebra

1. Prove or give a counterexample. If your answer includes a counterexample, you must explain why it works in order to receive full credit.

   (a) (2 points) Let R be an integral domain and I a non-zero ideal of R. Then R/I is a flat R-module.

   (b) (2 points) Let $n \geq 2$ be an integer. Then $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

   (c) (4 points) Let V be a finite-dimension vector space over a field F, and let $T : V \to V$ be an F-linear map. Assume that there exists a vector $v \in V$ such that

   $$V = \mathsf{Span}\{T^i(v) \mid i \in \mathbb{N}\}.$$

   If the minimal polynomial of T is $m_T(x) = (x-\lambda)^n$ for some $\lambda \in F$ then $\dim_F V = n$.

2. Let R be an integral domain with quotient field Q. We regard Q as an R-bimodule by defining the scalar multiplication operations as follows: for all $r \in R$ and $\frac{x}{y} \in Q$

   $$r \cdot \frac{x}{y} := \frac{rx}{y} \text{ and } \frac{x}{y} \cdot r := \frac{xr}{y}.$$

   Let N be a (left) R-module.

   (a) (2 points) Prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form

   $$\frac{1}{d} \otimes n,$$

   for some nonzero $d \in R$ and some $n \in N$.

   (b) (2 points) Let $U = R^\times = R \setminus \{0\}$, and define $U^{-1}N$ to be the set of equivalence classes of pairs $(u, n) \in U \times N$, under the relation

   $$(u, n) \sim (u', n') \quad \text{if and only if} \quad u'n = un' \text{ in } N.$$

   For a pair $(u, n) \in U \times N$, we write $\frac{n}{u}$ for the equivalence class of (u, n) under the above equivalence relation. The addition and scalar multiplication operations

   $$\frac{n_1}{u_1} + \frac{n_2}{u_2} := \frac{u_2 n_1 + u_1 n_2}{u_1 u_2}$$

   and

   $$r \cdot \frac{n}{u} := \frac{rn}{u}$$

   are well-defined, and they make $U^{-1}N$ into an R-module. Define a map $f : Q \times N \to U^{-1}N$ by

   $$f\left(\frac{a}{b}, n\right) = \frac{an}{b}, \quad \text{for } a \in R, b \in U, n \in N.$$

   Prove that this is a well-defined R-balanced map, and hence induces an R-module homomorphism from $Q \otimes_R N$ to $U^{-1}N$, denoted by F.

(c) (2 points) Show that the correspondence

$$\frac{n}{u} \to \frac{1}{u} \otimes n$$

from $U^{-1}N$ to $Q \otimes_R N$ is a well-defined R-linear map and is an inverse to the R-linear map F.

(d) (3 points) Use parts $(a) - (c)$ to show that Q is a flat R-module.

(e) (5 points) Assume that R is not a field. Show that $R \oplus Q$ is flat but is neither injective nor projective R-module.

3. (5 points) Let $A \in \mathbb{R}^{10\times 10}$ be a matrix with characteristic polynomial

$$c_A(x) = (x-1)^4 \cdot (x-2)^6.$$

Assume that

$$\mathsf{rank}\,(A - I_{10}) = 7, \quad \mathsf{rank}((A-I_{10})^2) = 6,$$

and

$$\mathsf{rank}\,(A - 2I_{10}) = 6, \quad \mathsf{rank}((A-2I_{10})^2) = 4, \quad \mathsf{rank}((A-2I_{10})^3) = 4.$$

Find the Jordan canonical form of A.

4. (5 points) Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be an F-linear map. Assume that the minimal polynomial of T coincides with its characteristic polynomial, and is given by

$$m_T(x) = p_1(x)^{d_1} \cdot \ldots \cdot p_\ell(x)^{d_\ell},$$

where $p_1(x), \ldots, p_\ell(x)$ are pairwise distinct irreducible polynomials over F, and $d_1, \ldots, d_\ell \geq 1$.

Let $v_1, \ldots, v_m \in V$ be m non-zero vectors in V, and define the subspaces

$$W_i := \mathsf{span}\{T^\ell(v_i) \mid \ell \in \mathbb{N}\}, \forall 1 \leq i \leq m.$$

Show that if $m \geq (d_1+1) \cdot \ldots \cdot (d_\ell + 1)$, then $W_i = W_j$ for some $i \neq j$.