# Qualifying Examination in Analysis

**May 2024**

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Solve all problems.

1. (a) Given $\sigma$-algebras $\mathcal{M}_1, \mathcal{M}_2$ on a set $X$, define the product $\sigma$-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$.

   (b) If $\mathcal{P}(X)$ denotes the power set of a set $X$, prove that $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N}^2)$, and if $\mu_c$ is the counting measure on $\mathbb{N}$ then $\mu_c \times \mu_c$ is the counting measure on $\mathbb{N}^2$.

   (c) Prove that if $\mathcal{B}_{\mathbb{R}}$ is the Borel $\sigma-$algebra on $\mathbb{R}$ then $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{P}(\mathbb{N})$ is the smallest $\sigma$-algebra on $\mathbb{R} \times \mathbb{N}$ containing sets of type $[a, b] \times \{n\}$, $a, b \in \mathbb{R}$, $a < b$, $n \in \mathbb{N}$, and

$$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{P}(\mathbb{N}) = \{\text{countable unions of sets of type } E \times \{n\},\ E \in \mathcal{B}_{\mathbb{R}},\ n \in \mathbb{N}\}$$

   [Hint: show first that the set on the right-hand side is a $\sigma$-algebra.]

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Explain what it means for $\mu$ to be continuous from below/above, and under which conditions they are true.

   (b) Recall that given a sequence $\{A_n\}_1^\infty \subseteq \mathcal{M}$ we have

$$\overline{\lim_n} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n = \{x \in X : x \in A_n \text{ for infinitely many } n\}$$

   Prove that if $\mu(X) < \infty$ then $\overline{\lim}_n \mu(A_n) \leq \mu\left(\overline{\lim}_n A_n\right)$.

   (c) Suppose that $\mu(X) < \infty$, and let $\{f_n\}$ be a sequence of measurable functions such that for some $a > 0$

$$\mu(\{x : |f_n(x)| \geq 1) > a > 0, \qquad \forall n \in \mathbb{N}$$

   prove that it's not possible that $f_n \to 0$ a.e.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) State the Lebesgue Dominated Convergence Theorem.

   (b) Consider the sequences of functions

$$f_n(x) = \frac{\sin(n^2x^2)}{n^2x^{3/2}}\, \chi_{(0,n]}(x), \qquad g_n(x) = \left(\sin\frac{x}{n}\right) \chi_{(0,n\pi/2]}(x).$$

   Establish if the following limits exist, in which case evaluate them:

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx, \qquad \lim_{n\to\infty} \int_0^\infty g_n(x)\, dx$$

4. (a) Define what it means for $T$ to be a bounded linear operator between normed vector spaces. For such an operator define $\|T\|$.

   (b) Consider the operator $T$ defined on the space of sequences $x = \{x_n\}_1^\infty$ as follows: $Tx = \{x_n + \lambda_n x_1\}_1^\infty$, where $\{\lambda_n\}_1^\infty$ is a complex-valued sequence such that $|\lambda_n| \le 2^{-n}$ for all $n \in \mathbb{N}$. Prove that $T$ is bounded from $\ell^1(\mathbb{N})$ to itself, and $\|T\| \le 2$.

   (c) Prove that if $\{\lambda_n\}_1^\infty = \{(-1)^n 2^{-n}\}_1^\infty$ then $\|T\| = 1$.

5. (a) Let $(X, \|\cdot\|)$ be a normed vector space. Explain in what sense $X$ can be identified with a subspace of $X^{**}$, and what it means for $X$ to be reflexive. Make sure notation and terms are reasonably well described.

   (b) Let $X$ be a reflexive normed vector space and denote the norm of $f \in X^*$ as $\|f\|$. Consider $f, g \in X^*$ and $\lambda, \mu \in \mathbb{R}$. Show that the following are equivalent:

      i. There exists $x_0 \in X$ with $\|x_0\| \le 1$ and $f(x_0) = \lambda, g(x_0) = \mu$.

      ii. For all $\alpha, \beta \in \mathbb{R}$ we have $|\alpha\lambda + \beta\mu| \le \|\alpha f + \beta g\|$.

   [Hint: For ii. $\Longrightarrow$ i. consider the map $\Phi_0 : \mathrm{span}\{f, g\} \to \mathbb{R}$ defined as $\Phi_0(\alpha f + \beta g) = \alpha\lambda + \beta\mu$.]

6. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Suppose that $\{f_n\}$ is a sequence of measurable functions on $(X, \mathcal{M}, \mu)$ such that $f_n \to f$ a.e., for some measurable $f$. Prove that for $0 < p \le \infty$ we have $\|f\|_p \le \underline{\lim}_n \|f_n\|_p$.

   (b) Find a sequence $\{f_n\}$ of Borel measurable functions on $\mathbb{R}$ such that

$$\|f\|_p < \underline{\lim}_n \|f_n\|_p < \overline{\lim}_n \|f_n\|_p,$$

   for any $p \in (0, \infty]$.