QUALIFYING EXAMINATION
MAY 2024

- Let S_n and A_n stand, respectively, for the permutation group and the alternating group on n letters.
- The symbol $H \lneqq G$ means that H is a subgroup of G, but H is not equal to G.
- Notation: $\mathbb{N} := \{0, 1, 2, ...\}$, $\mathbb{N}^+ := \{1, 2, ...\}$, $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$, $\mathbb{R}$:=the field of all real numbers.
- Every ring, in this exam, is assumed to have a multiplicative identity which is not equal to the additive identity.

ALGEBRA QUALIFYING EXAM

(I) Groups

(1) Prove that a group with 36 elements is not a simple group.
(2) Determine if the statement: "If $n \geq 3$, then A_n is generated by the three-cycles in S_n" is true or false. In either case, justify your answer.
(3) Let $H \lneq G$ where G is a finite group. Prove that there is an element $g \in G$ such that for each $x \in G$, $g \notin xHx^{-1}$.

(II) Rings

(4) Let $\mathcal{F}$ be a family of ideals in the ring R such that
   (i) $R \in \mathcal{F}$ and,
   (ii) for an element a of R and every ideal I of R, if $(I, a) \in \mathcal{F}$ and $I : a = \{r \in R : ra \in I\} \in \mathcal{F}$, then $I \in \mathcal{F}$ also.

   Let $\mathcal{G}$ be the complement of $\mathcal{F}$ in the collection of all ideals of R. Let J be an ideal in $\mathcal{G}$ which is maximal under the partial order on $\mathcal{G}$ given by inclusion. Prove that J is a prime ideal. (Note that J need not be a maximal ideal of R.)
(5) Let $p \in \mathbb{Z}$ be a prime and $f(X) := a_0 + a_1X + a_2X^2 + ... + a_{n-1}X^{n-1} + X^n \in \mathbb{Z}[X]$ be monic such that $\overline{f(X)} := \overline{a_0} + \overline{a_1}X + \overline{a_2}X^2 + ... + \overline{a_{n-1}}X^{n-1} + X^n$ is irreducible in $\mathbb{Z}_p[X]$ with $\overline{a_i} := a_i + p\mathbb{Z}$. Prove that $f(X)$ is irreducible in $\mathbb{Z}[X]$.
(6) Let $R := \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}^+$. Prove that R is semi-simple (as an R-module) if and only if n is square-free (i.e., p^2 does not divide n for every prime p).

(III) Fields

(7) Let F be a field and let $f(X) \in F[x]$ be an irreducible, separable polynomial of degree 3 with splitting field E. If a, b, and c are the roots of $f(X)$ in E and $b \notin F(a)$, compute the Galois group $Gal(E/F)$ of E over F.
(8) Let F be a field and let $f(X) \in F[x]$ with $g.c.d.(f(X), f'(X)) = 1$. Let E be the splitting field of $f(X)$. Prove that if all of the roots of $f(X)$ are conjugate under the Galois group $Gal(E/F)$, then $f(X)$ is irrreducible in $F[x]$. (Recall: *Roots of* $f(X)$ *are conjugate under* $Gal(E/F)$ means: if $u, v \in E$ are roots of $f(X)$, then there exists a $\sigma \in Gal(E/F)$ such that $\sigma(u) = v$.)

(IV) Modules and Linear Algebra

(9) Let k be a field and $g_1(X), g_2(X) \in k[X]$ with $g.c.d.(g_1(X), g_2(X)) = 1$. Let V be a finite-dimensional vector space over k and $T : V \to V$ be a linear operator with minimal polynomial $p_T(X) = g_1(X)g_2(X)$. For $i = 1, 2$, let $W_i := Ker(g_i(T))$ be T-invariant subspaces of V such that $V = W_1 \oplus W_2$.

For $i = 1, 2$, prove that $g_i(X)$ is the minimal polynomial for the restriction operator $T|_{W_i}$ of T to the subspace W_i.

(10) Let R be a commutative ring. If the sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of R-module homomorphisms is exact, prove that the sequence $L \otimes_R C \xrightarrow{f \otimes id_C} M \otimes_R C \xrightarrow{g \otimes id_C} N \otimes_R C \to 0$ of R-module homomorphisms is exact for every R-module C (i.e., prove that tensoring with C is right-exact).

(Hint: You may use the following two facts **without proving them**:

(i) The sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of R-module homomorphisms is exact if and only if the sequence $0 \to Hom_R(N, K) \to Hom_R(M, K) \to Hom_R(L, K)$ of R-module homomorphisms is exact for every R-module K.

(ii) "Adjoint Isomorphism" theorem: For R-modules U, V, W, there is an R-module isomorphism $Hom_R(U \otimes_R V, W) \cong Hom_R(U, Hom_R(V, W))$.)

(11) Let A be a 6 by 6 matrix with entries in $\mathbb{R}$ and characteristic polynomial $f_A(X) := (X^2 + 1)^2(X - 1)(X + 2)$.

(i) Write down all the possible minimal polynomials $p_A(X)$ for A.

(ii) Write down all (up to isomorphism) rational canonical forms of A over $\mathbb{R}$.