August 2023 Qualifying Examination

Read the whole test. The problems are not in any particular order of difficulty. There are three pages including this one.

If you have any question with the wording of any of the following problems please contact the supervisor immediately.

In what follows all rings R are assumed to have a multiplicative identity 1_R. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example. You may use a theorem as part of the solution of any problem, by quoting the theorem without proof. However, any theorem you quote should not be a statement that is essentially equivalent to the problem. Hints are only suggestions. It is not required that you use the hints.

1. Groups- 20 points

1. (4 points) How many non isomorphic groups are there of order 70? Prove your answer.

2. (4 points) Show that if G is a finite group of order n, then G is isomorphic to a subgroup of S_n.

3. (4 points) Show that if G is a group of order $2k$ and k is odd, then there is a subgroup of G of index 2.

4. (8 points) State True or False. If true, prove it and if false give a counter example and justify your example.
a. If a group has a proper subgroup of finite index, then it must have a proper normal subgroup of finite index.
b. If G is cyclic then the group of automorphisms of G is also cyclic.

2. Modules and Linear Algebra- 2+6+8+5+14 = 35 points.

1. Define a projective module.

2. Prove that the following are equivalent for a module P over a commutative ring R.
i. P is projective
ii. Every short exact sequence $0 \to N \to M \to P \to 0$ splits.
iii. There is a module N such that $P \oplus N$ is free.

3. Let R be a commutative ring.
a. Prove that free modules are projective but not conversely.

b. Prove that two if R modules M and N are projective then $M \oplus N$ and $M \otimes N$ are projective.

4. Let A be a square matrix over the rationals. Suppose that the characteristic polynomial of A factors as fg where f and g are non constant polynomials that are relatively prime in $\mathbb{Q}[x]$. Show that A is similar to a matrix $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B and C are matrices with characteristic polynomials f and g respectively.

5. Let K be an algebraically closed field and $M(n, K)$ be the vector space of all $n \times n$ matrices which is also a ring.

a. Prove that every matrix $A \in M(n, K)$ is similar to its transpose. What if K is not algebraically closed?

b. Prove that if V is a subspace of $M(n, K)$ of dimension ≥ 2, then there is a non zero singular matrix in V. What if K is not algebraically closed?

3. Rings- 20 points

State True or False. If true, prove it. If false give a counter example and justify your example.

a. The number of the units (finite or infinite) in a ring R is greater than or equal to the number of the nilpotent elements in R.

b. If R is a commutative ring then there are infinitely many maximal ideals in the polynomial ring $R[x]$.

c. The center of a simple ring is a field.

d. There are no fields with exactly 121 elements.

4. Fields- 25 points

1.(8points)

a. Let K be a field of characteristic p. Show that $x^p - x - a$ is either irreducible or factors into linear factors in $K[x]$.

b. Determine if $x^5 + 5x^4 - 20x^3 - 5x^2 - 6x + 9$ is irreducible in $Z[x]$.

2. (12 points)

a. Show that if L is a finite extension of K, then L must be algebraic over K.

b. Is the converse to a true? Prove your answer.

c. Prove or Disprove: If u is not algebraic over a field K then so is $f(u)$ for every polynomial $f \in K[x]$ of positive degree.

3. (5 points) Is it true that if f and g are two solvable polynomials over $\mathbb{Q}$, then so is fg. Prove your answer. Answer the same for $f + g$.