Qualifying Examination in Analysis

May 2023

- If you have any difficulty with the wording of the following problems please contact the supervisor immediately.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Solve all problems.

1. Let μ be a measure on a measure space $(X, \mathcal{S})$ and let $\{E_j\}_1^\infty \subseteq \mathcal{S}$, be so that $\mu(E_j \cap E_k) = 0$ for all $j \neq k$. Let $N = \bigcup_{j,k} E_j \cap E_k$.
   (a) Prove that $\mu(E) = \mu(E \setminus N)$.
   (b) Show that the sets $\{E_j \setminus N\}$ are disjoint.
   (c) Prove that $\mu(E) = \sum_1^\infty \mu(E_j)$.

2. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Suppose that $L^2(X)$ is a subset of $L^1(X)$.
   (a) Show that there is a constant C such that for all f in $L^2(X)$ we have $\|f\|_1 \leq C\|f\|_2$. [Hint: use the closed graph theorem.]
   (b) Show that $\mu(X) < \infty$.
   (c) Prove that $L^p(X)$ embeds continuously in $L^q(X)$, if $1 \leq q < p < \infty$.

3. (a) Show that for every $f \in L^1(\mathbb{R})$ and any $\epsilon > 0$ we have

$$\lim_{n\to\infty} n^{-\epsilon} f(nx) = 0$$

   for almost all $x \in \mathbb{R}$. [Hint: Consider a series.]
   (b) Construct a function $f \in L^1(\mathbb{R})$ such that f is unbounded on a set of positive measure inside $[N, \infty)$, for any $N \in \mathbb{N}$.

4. On $[0,\infty)$ consider the sequence of functions

$$f_n(x) = \frac{1-|\cos x|^n}{(1+\frac{x}{n})^2} - \frac{1-\cos\left(\sqrt{\frac{x}{n}}\right)}{x^{3/2}}$$

Show that $\|f_n\|_1 < \infty$ for all $n = 1, 2, \ldots$, then show that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = +\infty.$$

5. Let X be an infinite dimensional normed space and let $B_X := \{x \in X : \|x\| \le 1\}$ be the closed unit ball of X.

   (a) Let F be a finite dimensional subspace of X and let $w \in X \setminus F$. Pick $z \in F$ such that $0 < \|w - z\| < \frac{3}{2}\operatorname{dist}(w, F)$ and define $x := \frac{w-z}{\|w-z\|}$. Prove that

   $$\operatorname{dist}(x, F) > \frac{2}{3}.$$

   (b) Starting with $x_1 \in X$ with $\|x_1\| = 1$ and $F_1 = \{0\}$ find inductively points x_n in X with $\|x_n\| = 1$ and finite-dimensional subspaces $F_n = \operatorname{span}(x_1, \ldots, x_{n-1})$ of X related as in part (a). Prove that $\frac{3}{4}x_n + \frac{1}{4}B_X$, $n = 1, 2, \ldots$ are mutually disjoint sets.

   (c) Show that there is no positive measure μ on a σ-algebra that contains the open balls of X with the following properties:

      i. μ is translation invariant, i.e., $\mu(x + A) = \mu(A)$ for all $x \in X$ and any measurable set $A \subset X$.

      ii. $0 < \mu(B) < \infty$ for any open ball B in X.

6. Let $E = \{x = (x_n)_{n=1}^\infty \mid x_{2k} = 0,\ k = 1, 2, \ldots\}$. Prove the following:

   (a) E is closed linear subspace of ℓ^2 and of ℓ^1.

   (b) For every bounded linear functional $f : E \to \mathbb{R}$ and any $\epsilon > 0$ construct infinitely many linear extensions $F : \ell^1 \to \mathbb{R}$ of f with norm $\|F\| \le (1+\epsilon)\|f\|$.

   (c) Every bounded linear functional $g : E \to \mathbb{R}$ has unique extension $G : \ell^2 \to \mathbb{R}$ with the same norm. [Hint: Use a version of the Riesz representation theorem.]

7. Either find an example or disprove each of the following statements:

(a) There is a Lebesgue measurable subset A of $[0,1]\times[0,1]$ such that

$$|\{z\in[0,1]:\ (x,z)\in A\}| = x \quad \text{and} \quad |\{z\in[0,1]:\ (z,y)\in A\}| = 1-y$$

for all x and y in $[0,1]$.

(b) There is a Lebesgue measurable subset B of $[0,1]\times[0,1]$ such that

$$|\{z\in[0,1]:\ (x,z)\in B\}| = x^2 \quad \text{and} \quad |\{z\in[0,1]:\ (z,y)\in B\}| = 1-y^2$$

for all x and y in $[0,1]$.

8. Consider the complex measure

$$\nu(A) = \int_A e^{ix-|x|}dx$$

defined on Lebesgue measurable subsets A of $\mathbb{R}$.

(a) What is the variation measure $|\nu|$ and the total variation $\|\nu\|$?

(b) Prove that if A has positive Lebesgue measure, then for any $F_1,\ldots,F_N$ disjoint measurable subsets of A we have

$$\sum_{k=1}^{N} |\nu(F_k)| < |\nu|(A).$$