May 2023 Qualifying Examination

Read the whole test. The problems are not in any particular order of difficulty. There are three pages including this one.

If you have any question with the wording of any of the following problems please contact the supervisor immediately.

In what follows all rings R are assumed to have a multiplicative identity 1_R. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example. You may use a theorem as part of the solution of any problem, by quoting the theorem without proof. However, any theorem you quote should not be a statement that is essentially equivalent to the problem. Hints are only suggestions. It is not required that you use the hints.

1. Groups- 21 points

1. Prove or give a counter example with proper justification. (4 points each) ($Z(G)$ denotes the center of G.)

a. All groups of order 80 are solvable.

b. If a finite group G is of odd order, then there is at least one element in G which has no square root. ($g \in G$ is said to have a square root if there is an element $h \in G$ such that $h^2 = g$.)

c. For any two subgroups H and K of G, HK is a subgroup of G if and only if $HK = KH$.

d. For any finite non abelian group G, $|Z(G)| \leq \frac{1}{4}|G|$.

2. (5 points) G is a group and H is a normal subgroup of G which is not contained in $Z(G)$. If $|H| = 3$, prove that G has a subgroup of index 2.

2. Modules and Linear Algebra- 5+7+7+10= 29 points.

1. R is a commutative ring. M is an R module which has no proper submodules other than $\{0\}$. Show that there is a prime ideal p of R such that $M \cong R/p$.

2. R is a commutative ring. Prove that two R modules M and N are injective if and only if $M \oplus N$ is injective.

3. R is a commutative ring and M is a finitely generated R-module and $\phi : M \to M$ is an R-module homomorphism. Show that if ϕ is surjective then it is an isomorphism.

(Hint: Find an appropriate matrix over a ring R to use Cayley Hamilton Theorem, namely, the matrix A satisfies the characteristic polynomial $\det(xI - A)$.)

4. Prove or give a counter example with proper justification. (5 points each)

a. For any two $n \times n$ matrices A, B over a field k, the eigenvalues of AB are the same as the eigenvalues of BA.

b. For any matrix A over real numbers, the eigenvalues of AA^T must be non negative. (A^T denotes the transpose of A)

3. Rings- 24 points

Prove or give a counter example.

a. Every polynomial $f \in R[x]$ of positive degree has at most $d = \deg f$ roots in R if and only if R is an integral domain.

b. If an element in a ring R has a left inverse but is not a unit then it must have infinitely many left inverses.

c. If $f, g \in R[x]$ are two monic polynomials of positive degree over a commutative ring R, there are $q, r \in R[x]$ such that $f = qg + r$, where $r = 0$ or $\deg r < \deg g$.

d. A commutative ring R is a Principal Ideal Domain if and only if every prime ideal is principal.

4. Fields- 6+6+7+7=26 points

1. Let k be a field and $f \in k[x]$ be of degree n. Show that there is a field E containing k such that $f(x)$ factors into linear factors in $E[x]$ and that $[E : k] \leq n!$

2. Prove that if K is an infinite field of characteristic p and $F = K(u, v)$ is an extension of degree p^2 such that $u^p, v^p \in K$ then there is an infinite number of intermediate fields $F \supset L \supset K$.

3. K is a field of characteristic p. Show that $x^p - x - a \in K[x]$ is either irreducible or factors in to linear factors in $K[x]$. Conclude from this (or otherwise) that $x^p - x - 1$ is irreducible over $\mathbb{Q}$ for all prime numbers p.

4. Is it true that if f and g are two polynomials solvable by radicals over $\mathbb{Q}$, then so is fg? Prove your answer. Answer the same for $f + g$.