August 2022 Qualifying Examination

Read the whole test. The problems are not in any particular order of difficulty. There are three pages including this one.

If you have any question with the wording of any of the following problems please contact the supervisor immediately.

In what follows all rings R are assumed to have a multiplicative identity 1_R. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example. You may use a theorem as part of the solution of any problem, by simply quoting the theorem without proof. However, theorem you quote should not be a statement that is essentially equivalent to the problem. Hints are only suggestions. It is not required that you use the hints.

In any multiple part question, you may use the earlier part of the question to answer the later part even if you did not solve the earlier part.

I. Groups- 21 points

Prove or give a counter example with proper justification.

a. Every finite group is isomorphic to a subgroup of A_n for some n.

b. Any two groups of order 35 must be isomorphic to each other.

c. If G has a subgroup of finite index, then it has a normal subgroup of finite index.

d. G is solvable if and only if both H and G/H are solvable for any normal subgroup H of G.

II. Modules and Linear Algebra- 30 points.

1. (6 points) Show that for any matrix A over real numbers, AA^T has the same rank as A.

2. (9 points) a. Define Projective and Injective modules over a ring R.
b. Is $\mathbb{Q}$ a projective $\mathbb{Z}$ module? Prove your answer.

c. Is $\mathbb{Q}$ an injective $\mathbb{Z}$ module? Prove your answer.

3. (8 points) A is an $n \times n$ matrix over $\mathbb{Q}$.

a. Show that if λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.
b. Show that if A is nilpotent, then zero is the only eigenvalue of A.
c. Show that if A is nilpotent, then $tr(A^t) = 0$ for all positive integers t.
d. Show that if $tr(A^t) = 0$ for all positive integers t, then A must be nilpotent.

4. a. (2 point) State the Structure theorem for finitely generated modules over a Principal Ideal Domain.
b. (5 points) Prove that a proper submodule of a free R- module must be free if and only if R is a Principal ideal domain.

III. Rings- 6+6+7+6 points

1. $K \subset L$ are fields. $f, g \in K[x] \subset L[x]$.
Is it rue that if f and g are relatively prime in $L[x]$, they are also relatively prime in $K[x]$.

2. a. Show that if a monic polynomial $f(x) \in \mathbb{Z}[x]$ is irreducible modulo p for some prime number p, then $f(x)$ must be irreducible in $\mathbb{Z}[x]$
b. What is the converse of a? Is it true? Justify your answer.

3. R is a ring. *Recall that an additive subgroup I of R is a left ideal if $x \in I, r \in R \implies rx \in I$ and that a maximal left ideal of R is a left ideal of R that is not contained in any other proper left ideal.*

a.Use Zorn's lemma to show that every proper left ideal of R is contained in a left maximal ideal.
b. Show that the intersection of all non-zero left ideals of R is a two sided ideal.

4. *A commutative ring R is called a LOCAL ring if it has exactly one maximal ideal.*
Show that a commutative ring R is a local ring if and only if the set of non-units in R form an ideal.

IV. Fields- 6+6+6+6 points

1. K is a field and let $L = K(e)$ where e is transcendental over K. Let $u \in L$. Show that either $u \in K$ or L is algebraic over $K(u)$.

2. Let $f(x) = x^4 + 4x^2 + 3$ Let L be the Splitting field of f over $\mathbb{Q}$. Find the Galois group $G = G(L/\mathbb{Q})$. What is its order? If this is isomorphic to a familiar group, say what it is and why. You should justify your answers with reasons.

3 . Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5 which has exactly two real roots. Show that the Galois group of f is S_5.

4. Let F be a finite field.
a. Show that the characteristic of F is a prime number p.
b. Show that $|F| = q$, where q is a power of p.
c. If a is algebraic over F, then show that $a^{q^m} = a$ for some non negative integer m.