Analysis Qualifying Exam - August 2022

Work through all parts. Your work will be graded for correctness, completeness, and clarity.

Note: *Below $\mathcal{L}$ denotes the class of Lebesgue measurable sets in $\mathbb{R}^n$, m the Lebesgue measure on $(\mathbb{R}^n, \mathcal{L})$, $|E| = m(E)$.*

1. For f in $L^1_{\text{loc}}(\mathbb{R}^n)$ define the following versions of the Hardy-Littlewood maximal function:

$$M_1 f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dm(y), \qquad M_2 f(x) = \sup_{B_x} \frac{1}{|B_x|} \int_{B_x} |f(y)| dm(y)$$

$$M_3 f(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| dm(y)$$

where $B(x,r)$ is the open ball with center x and radius r, $Q(x,r)$ is the (closed) cube centered at x with edges parallel to the coordinate axes and with length $2r$, and where B_x is any open ball containing x.

(a) Let $n = 2$ and find explicit constants $a, b, c, d > 0$ such that

$$aM_1 f(x) \le M_3 f(x) \le bM_1 f(x), \qquad cM_1 f(x) \le M_2 f(x) \le dM_1 f(x), \qquad \forall x \in \mathbb{R}^2.$$

(Note: this is also true in $\mathbb{R}^n$ – when $n = 2$ computations are slightly easier).

(b) State the Hardy-Littlewood Theorem (about the maximal function) in $\mathbb{R}^n$, and explain why in $\mathbb{R}^2$ the theorem works for any of the above versions of the H-L maximal function (note: this is also true in $\mathbb{R}^n$).

(c) Show that if $M_2 f(x_0) > \alpha$ then there is $\delta > 0$ such that $M_2 f(x) > \alpha$ when $|x - x_0| < \delta$.

2. (a) Show that the space $B([0,\infty]) = \{f : [0,\infty) \to \mathbb{C}, \text{ bounded, measurable}\}$ is an inner product space under

$$(f,g) = \int_{[0,\infty)} \frac{f(x)\overline{g(x)}}{1+x^3} dm(x).$$

(b) Show that the space $\big(B([0,\infty]), (\cdot,\cdot)\big)$ is not a Hilbert space.

(c) Let $\{e_j\}_1^\infty$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, endowed with the natural inner product that makes it a Hilbert space (over $\mathbb{C}$). Show that if $e_{jk}(x,y) = e_j(x)e_k(y)$, for $x, y \in \mathbb{R}^n$ and $j, k \in \mathbb{N}$ then $\{e_{jk}\}_{j,k\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^{2n})$ (with its natural inner product).

3. (a) Define the concept of bounded linear functional on a normed vector space $(X, \|\cdot\|)$. Define the dual X^* of the space X.

(b) State the Hahn-Banach Theorem on normed vector spaces.

(c) Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on X. Show that $\|(x,y)\| := \|x\|_1 + \|y\|_2$ $(x, y \in X)$, defines a norm on the vector space $X \times X$.

(d) In the notation of 3(c) above, suppose that f is a linear functional on X such that $|f(x)| \le \|(x,x)\|$ for all $x \in X$. Show that there exist f_1, f_2 linear functionals on X such that $f = f_1 + f_2$ and $|f_1(x)| \le \|x\|_1$, $|f_2(x)| \le \|x\|_2$, for all $x \in X$.

4. (a) Define what it means for a measure space $(X, \mathcal{M}, \mu)$ to be σ-finite.

   (b) Show that if $(X, \mathcal{M}, \mu)$ is σ-finite and $E \in \mathcal{M}$ is so that $\mu(E) > 0$, then there is $F \in \mathcal{M}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$. With an example show that this conclusion may fail without the σ-finiteness hypothesis.

   (c) Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space and let $f \in L^1(\mu)$, and real-valued. Define the set of real numbers

$$A_f := \left\{ \frac{1}{\mu(E)} \int_E f \, d\mu \ : \ E \in \mathcal{M}, \, 0 < \mu(E) < \infty \right\}.$$

   Prove that if $A_f \subseteq [a, b]$, then $a \leq f(x) \leq b$ for almost every $x \in X$.
   [Hint: argue by contradiction.]

5. (a) Let X, Y be normed spaces, let $T_n, T : X \to Y$ be bounded linear operators, and $x_n, x \in X$. Show that if $T_n \to T$ (in the operator norm topology) and $x_n \to x$, then $T_n x_n \to Tx$.

   (b) Let X, Y be normed spaces and let $T : X \to Y$ be linear with the following property: for each sequence $\{x_n\}$ in X with $x_n \to 0$ the sequence $\{Tx_n\}$ is bounded in Y. Show that T is bounded. [Hint: argue by contradiction.]

6. In this problem you can assume the validity of the following formula (stated in the May 2022 qualifying exam):

$$\int_{\mathbb{R}^n} f \, g \, dm = \int_0^\infty \left(\int_{\{x \in \mathbb{R}^n : f(x) \geq t\}} g(x) dm(x) \right) dt \tag{1}$$

valid for measurable $f, g : \mathbb{R}^n \to [0, \infty)$.

Let $f \in L^1(\mathbb{R}^n, m)$ satisfy the following property:

$$\int_A |f| \, dm \leq \sqrt{m(A)}, \quad \text{for all } A \in \mathcal{L} \text{ such that } m(A) < \infty. \tag{2}$$

   (a) Show that for every $t > 0$, we have $m\{x \in \mathbb{R}^n : |f(x)| \geq t\} \leq t^{-2}$.

   (b) Prove that if $1 < p < 2$, then f is in $L^p(\mathbb{R}^n, m)$.

   (c) Provide an example of a function $f \in L^1(\mathbb{R}, m)$ satisfying (2) and that it is not in $L^2(\mathbb{R}, m)$.