Analysis Qualifying Exam - May 2022

Work through all parts. Your work will be graded for correctness, completeness, and clarity.

Note: *Below $\mathcal{L}$ denotes the class of Lebesgue measurable sets in $\mathbb{R}^n$, m the Lebesgue measure on $(\mathbb{R}^n, \mathcal{L})$, and "nonmeasurable" means "not in $\mathcal{L}$".*

1. (a) Define what it means for a function $f : [0, 1] \to \mathbb{R}$ to be of bounded variation.

   (b) Give an example of a sequence $\{f_n\} \subset BV[0, 1]$ such that $f_n \to f$ pointwise on $[0, 1]$, yet f is not of bounded variation.

   (c) Explain why the characteristic function of the Cantor set is singular on $[0, 1]$ and not of bounded variation on $[0, 1]$.

2. (a) Formulate Tonelli's theorem for functions defined on intervals (bounded or not) in $\mathbb{R}^{n+m}$.

   (b) Prove that there is no Borel set $A \subseteq [0, 1] \times [0, 1]$ such that A_x is countable for all $x \in [0, 1]$ and $[0, 1] \setminus A^y$ is countable for all $y \in [0, 1]$. [Here $A_x = \{y \in [0, 1] : (x, y) \in A\}$, etc.]

   (c) Use Tonelli's theorem to derive the following formula: if $f, g : \mathbb{R}^n \to [0, \infty)$ are measurable functions on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f\, g\, dm = \int_0^\infty \left(\int_{\{x \in \mathbb{R}^n : f(x) \geq t\}} g(x)\, dm(x) \right) dt.$$

   (d) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $m(\{x \in \mathbb{R}^n : |f(x)| \geq t\}) \leq t^{-2}$ for all $t > 0$. Prove that there exists a constant $C > 0$ such that for any measurable subset G of $\mathbb{R}^n$ we have

$$\int_G |f|\, dm \leq C \sqrt{m(G)}.$$

   [Hint: Use c) and split the integral in the t variable appropriately.]

3. (a) Explain what it means for a measure space to be complete.

   (b) Prove that there exists a nonmeasurable subset of $\mathbb{R} \setminus \mathbb{Q}$ (you can use without proof that there exist nonmeasurable sets in $\mathbb{R}$).

   (c) Let $\mu = \sum_{r \in \mathbb{Q}} \delta_r$ and $\nu = m_{/(-\infty,0]} + \mu_{c/(0,\infty)}$, where δ_a is the Dirac's delta at a and μ_c is the counting measure. Establish whether or not $(\mathbb{R}, \mathcal{L}, \mu)$, $(\mathbb{R}, \mathcal{L}, \nu)$ are complete.

   (d) Let $Q = (0, 1) \times (0, 1) \subseteq \mathbb{R}^2$, and let $N \subseteq (0, 1) \subseteq \mathbb{R}$ be a nonmeasurable set. Prove that $E = Q \cup \{(x, 0),\ x \in N\}$ is Lebesgue measurable.

   (e) Prove that E is not Borel measurable.

4. (a) Define what it means for a sequence of functions $\{f_n\}$ on $(\mathbb{R}, \mathcal{L}, m)$ to converge in measure to a function f.
   (b) Here and in c), d) below let $f_n(x) = (\sin x)^n$. Show that $\{f_n\}$ converges to 0 a.e. in $\mathbb{R}$.
   (c) Using the definition show that $\{f_n\}$ does not converge in measure to 0.
   (d) Using the definition show that $\{f_n \mathcal{X}_{[0,2\pi]}\}$ converges to 0 in measure.

5. On $(\mathbb{R}, \mathcal{L}, m)$ consider the measures

$$\nu = m_{/(-\infty,0]} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \delta_k, \qquad \mu = m + 2\delta_1 + 3\delta_2.$$

   (a) Explain why ν defines a signed measure, and why it is σ-finite.
   (b) Write the Hahn decomposition of $\mathbb{R}$ with respect to ν.
   (c) Write the Lebesgue-Radon-Nikodym decomposition of ν with respect to μ.
   (d) Compute the Radon-Nikodym derivative, with respect to μ, of the component of ν which is absolutely continuous with respect to μ.
   (e) Show that on $(\mathbb{R}, \mathcal{L})$ we have $m \ll \mu_c$, the counting measure, however there cannot exist a nonnegative μ_c-integrable f such that $dm = f d\mu_c$.

6. (a) (i) Show that the operator $Tf(x) = \displaystyle\int_{[0,x]} e^t f(t) dm(t)$ is well-defined and continuous from $L^1([0,1])$ to $L^\infty([0,1])$, both spaces being equipped with their natural norms. (ii) Compute the norm of T.
   (b) Let $C[0,1]$ be equipped with the uniform norm. (i) Show that the operator $Tf = |f|^{1/2}$ is well defined from $C[0,1]$ to itself, and it is continuous. (ii) Show that T is not bounded, in the sense that there is no C such that $\|Tf\|_u \le C\|f\|_u$ for all $f \in C[0,1]$. (iii) How is this not in contradiction with the general theorem about bounded operators and continuity?
   (c) Explain in what precise sense a normed space X is identified with a subspace of its double dual X^{**}.
   (d) Give the definition of reflexive normed space and give one example (infinite dimensional, no proof needed).