May 2022 Qualifying Examination

Read the whole test. The problems are not in any particular order of difficulty. There are three pages including this one.

If you have any question with the wording of any of the following problems please contact the supervisor immediately.

In what follows all rings R are assumed to have a multiplicative identity 1_R. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example. You may use a theorem as part of the solution of any problem, by simply quoting the theorem without proof. However, theorem you quote should not be a statement that is essentially equivalent to the problem. Hints are only suggestions. It is not required that you use the hints.

1. Groups- 20 points

Prove or give a counter example with proper justification.

a. Any non trivial subgroup of a simple group must be simple.

b. An infinite group must have an infinite number of subgroups.

c. Any Subgroup of finite index of a finitely generated group must be finitely generated.

d. There are no nonabelian groups of order 2023.

2. Modules and Linear Algebra- 5+5+10+10=30 points.

1. Show that for any matrix A over real numbers, AA^T has the same rank as A.

2. N is a submodule of a module M over a commutative ring R. If both N and M/N are finitely generated show that M is also finitely generated.

3. Show that the following are equivalent for a commutative ring R.
i. Every R module is projective.
ii. Every R module is injective.
iii. Every short exact sequence of R modules is split.

4. Prove or give a counter example:
a. If an $m \times n$ matrix A with entries in a commutative domain R is of rank 1, then then there are $m \times 1$ and $1 \times n$ matrices X and Y over R such that $A = XY$.

b. A square matrix A is diagonalizable over a field k if and only if its minimal polynomial has no repeated roots.

3. Rings- 24 points

a. Prove that a polynomial $f \in R[x]$ of degree d has at most d roots in R if and only if R is an integral domain.

b. R is a ring. Recall that a maximal left ideal of R is a left ideal of R that is not contained in any other proper left ideal.
Show that the intersection of all maximal left ideals of R is a two sided ideal.

c. If F is a finite field, show that every element $c \in F$ can be written as $c = a^2 + b^2$ for some $a, b \in F$.

d. Prove or Disprove: A commutative ring $R \neq \{0\}$ is a field if and only if every proper ideal in R is a prime ideal.

4. Fields- 6+7+6+7=26 points

a. $K \subset L \subset M$ are fields. Prove that if L is algebraic over K and M is algebraic over L, then M is algebraic over K.

b. A field extension M/K is said to be a simple extension if $M = K(a)$ for some $a \in M$. Show that if there are only finitely many intermediate fields L with $K \subset L \subset M$, then M/K is a simple extension. (Hint: Show that M/K must be a finite extension. Even if you cannot, assume that M/K is finite and show that it is simple for partial credit.)

c. Give a specific example of a polynomial $f \in \mathbb{Q}[x]$ that is not solvable by radicals. Justify your example. (Hint: Try something like $x^5 - 5x + \ldots$)

d. Suppose K is a field containing all nth roots of 1. Let M be the splitting field over K of $x^n - a$ for some $a \in K$. Show that the Galois group $G(M/K)$ is cyclic of order a factor of n.