Qualifying Examination

(*August 2018*)

- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.
- Let n denote a positive natural number, i.e., $n \in \mathbb{N}^+$.
- Let S_n denote the symmetric group on n letters.
- Let $\mathbb{Z}$ denote the group or ring of integers with the usual operations and let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the groups or fields of rational, real, and complex numbers, respectively, with the usual operations.
- All rings are assumed to be commutative with identity. The set of all prime ideals in R is denoted by $Spec(R)$.
- Let K denote a field and $\mathcal{M}_{n\times n}(K)$ denote the set of all $n \times n$ matrices with entries in K. Let $(V, \mathcal{B}, f)$ stand for the n-dimensional K-vector space V with basis $\mathcal{B}$ and bilinear form $f : V \times V \to K$. Let $[f]_{\mathcal{B}}$ be the representing matrix of the bilinear form f on V with respect to the basis $\mathcal{B}$. Let I denote the $n \times n$ identity matrix over the field K.

Algebra Qualifying Exam

(I)Groups

(**1**) Let H be a subgroup of the group G and let $a \in H$.
(**a**) (1*point*) Define the centralizer $C_G(a)$ of a in G and the centralizer $C_H(a)$ of a in H.
(**b**) (4*points*) Let G be a finite group and $[G : H] = 2$. Let $m := |\{gag^{-1} : g \in G\}|$ and $n := |\{hah^{-1} : h \in H\}|$. Prove that either $n = m$ or $n = m/2$.

(**2**) Let G be a finite group and let $Z(G)$ denote the center $\{g \in G : ga = ag$ for all $a \in G\}$ of G.
(**a**) (2*points*) Carefully state the class equation for groups.
(**b**) (3*points*) Let p be a prime such that p divides $|G|$. Suppose that, for each $x \in G \setminus Z(G)$, the prime p does not divide $|C_G(x)|$. Prove that p divides $|Z(G)|$.
(**c**) (2*points*) Prove Cauchy's Theorem. You may assume (i.e., do not prove) Cauchy's Theorem for the case of all abelian groups.

(**3**) (3*points*) Determine if the following statement is true or false and *substantiate* your answer. There is no simple group of order 30.

(**4**) Let G' stand for the commutator subgroup of the group G. Determine if the following statement is true or false and *substantiate* your answer.
(**a**) (2*points*) Let H be a normal subgroup of G. If G/H is an abelian group, then $G' \leq H$.
(**b**) (3*points*) If $G' = G$ and $G \neq \{1_g\}$, then G is solvable.

(II)Rings

(**5**) (5*points*) Let R be a principal ideal domain and let $\{a_i : i \in \mathbb{N}^+\}$ be a subset of non-zero, non-unit elements of R such that a_{i+1} divides a_i for each $i \in \mathbb{N}^+$. Prove that there exists an $n \in \mathbb{N}^+$ such that $(a_i) = (a_n)$ for all $i \geq n$. (Since this statement implies that principal ideal domains are noetherian rings, your proof should not invoke the noetherian property.)

(**6a**) (5*points*) State Eisenstein's Criterion.
(**6b**) (5*points*) Determine if the following statement is true or false and *substantiate* your answer. The polynomial $f(X) = 3(X-1)^3 - 8(X-1)^2 + 4(X-1) + 2$ is irreducible in $\mathbb{Q}[X]$.

(**7**) (5*points*) Let S be a multiplicatively closed subset of the ring R. Determine if the following statement is true or false and *substantiate* your answer.
The ring homomorphism $\phi : R \to S^{-1}R$, given by $\phi(r) = r/1_R$ is a monomorphism.

(III)Fields

(**8**) (5*points*) Let F be a field extension of the field K with $u \in F$ algebraic over K. Prove that there exists a monic, irreducible polynomial $f(X) \in K[X]$ such that $f(u) = 0$ and such that $K[u]$ is ring-isomorphic to $K[X]/(f(X))$.

(**9**) Let p be a prime, $n \in \mathbb{N}^+, q := p^n$, and F be a splitting field of $f(X) := X^q - X$ over the prime field $\mathbb{Z}_p$. Let $\mathcal{F}_0 := \{u \in F \mid f(u) = 0\} = \{u_1, u_2, \ldots, u_t\}$ with $\mid \mathcal{F}_0 \mid = t$.
(**a**) (3*points*) What is t, i.e., how many distinct roots does $f(X)$ have? (Substantiate your answer.)
(**b**) (5*points*) Show that $\mathcal{F}_0$ is a field.

(**10**) (7*points*) Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ be the splitting field of $f(X) = (X^2 - 2)(X^2 - 3) \in \mathbb{Q}[X]$ over $\mathbb{Q}$. Compute the Galois group $Aut_{\mathbb{Q}}(E) = Gal(E/\mathbb{Q})$.

(IV)Modules

(**11**) Let R be a ring. Determine if the following statement is true or false and *substantiate* your answer.
(**a**) (5*points*) Every R-module M is a submodule of a projective R-module.
(**b**) (5*points*) If G is a non-zero, abelian group, then $G \otimes_{\mathbb{Z}} G \neq 0$.
(**c**) (5*points*) Let L, M, N be three non-zero R-modules such that $Hom_R(L, N) = 0$. Then $Hom_R(L \otimes_R M, N) = 0$.

(**12**) (5*points*) Let R be a principal ideal domain and E be a divisible R-module. Prove that E is an injective R-module.

(V)LinearAlgebra

(**13**) (5*points*) State and prove Riesz' Representation Theorem.

(**14**) (5*points*) Determine if the following statement is true or false and *substantiate* your answer. Let $(V, \mathcal{B}, f)$ be a real, bilinear form such that

$$[f]_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then f has an orthonormal basis.

(**15**) (5*points*) Substantiate fully your answers in the following. Let $A \in \mathcal{M}_{5\times 5}(\mathbb{R})$ be of minimal rank with respect to having $\{-1, 0, 1\}$ as its set of all characteristic values. Write down (up to similarity) all the possible Jordan Canonical Forms of A.

(**16**) (5*points*) Let $A \in \mathcal{M}_{n\times n}(K)$ such that $n \geq 2, A^n = 0_{n\times n}$, and $A^{n-1} \neq 0_{n\times n}$. Prove that there does not exist a matrix $B \in \mathcal{M}_{n\times n}(K)$ such that $B^2 = A$.