MU DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM; MAY 2016

This exam consists of eleven questions in two pages. Your answers will be graded for correctness, completeness, and clarity. Points will be subtracted for incorrect arguments, arguments based on incorrect assumptions, and arguments not pertinent to the questions. m denotes the standard Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$. "Almost always" is synonymous to "almost everywhere."

(1) For $(X, \mathcal{M}, \mu)$ measure space and f in $L^p(\mu) \cap L^\infty(\mu)$ for some $p < \infty$, show that f is in $L^q(\mu)$ for all $q > p$. Then show that $\|f\|_p \to \|f\|_\infty$, as $p \to \infty$.

(2) For any sequence of positive measurable functions f_n, $n \in \mathbb{N}$ on measure space $(X, \mathcal{M}, \mu)$, show that $\displaystyle\int_X \liminf_{n\to\infty} f_n(x)\,\mu(dx) \le \liminf_{n\to\infty} \int_X f_n(x)\,\mu(dx)$. You may use the Monotone Convergence Theorem without proof.

(3) Let $f : [a,b] \to \mathbb{R}$ be Borel measurable. Show that for any $\epsilon > 0$ there is K compact subset of $[a,b]$ such that $m\left([a,b] \setminus K\right) < \epsilon$ and the restriction of f on K is continuous. You may use appropriate approximations and Egorov's Theorem (on uniform convergence on the complement of an appropriate set) without proof.

(4) For $(X, \mathcal{M}, \mu)$ a measure space and f positive and measurable on X show that the set $\{(x,t) : 0 < t < f(x)\}$ is in the product σ-algebra $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$. Then show that $\displaystyle\int_X f(x)\ \mu(dx) = \int_{(0,\infty)} \mu\left(\{x : f(x) > t\}\right)\ m(dt)$. You may use the Fubini-Tonelli theorem without proof.

(5) For σ the standard measure on the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, show that $\sigma(S^{n-1}) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$.

(6) For $f \in L^1(\mathbb{R}^n)$ show that $\lim_{r\to 0} \dfrac{1}{m(B_r)} \displaystyle\int_{B_r(x)} f(y)m(dy) \to f(x)$ for almost all x. You may use without proof the Maximal Theorem for the Hardy-Littlewood maximal function and the approximation in L^1 by continuous functions.

(7) On a separable, infinite dimensional Hilbert space describe all translation invariant Borel measures that are finite on bounded balls. Provide a complete justification of your answer.

(8) Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone then f is Borel measurable.

(9) If $\mathcal{A}$ is an algebra and $\mathcal{C}(\mathcal{A})$ is the monotone class generated by $\mathcal{A}$, for E in $\mathcal{C}(\mathcal{A})$ let $\mathcal{C}_E = \{F \in \mathcal{C}(\mathcal{A}) : E \setminus F, F \setminus E, \text{ and } E \cap F \text{ are all in } \mathcal{C}(\mathcal{A})\}$. Show that $\mathcal{C}(\mathcal{A}) \subset \mathcal{C}_F$, for all F in $\mathcal{C}(\mathcal{A})$.

(10) Let ν be signed measure on $(X, \mathcal{M})$ with $\nu(E) < +\infty$ for all E in $\mathcal{M}$. For P_i positive sets for ν with $\nu(P_i) \to \sup\{\nu(E) : E \text{ positive for } \nu\}$, set $P = \bigcup_i P_i$. Show that $X \setminus P$ is a negative set for ν.

(11) Show that if a Hilbert space H has a countable orthonormal basis then every orthonormal basis of H is countable.