MU DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM; AUGUST 2016

This exam consists of eleven questions in two pages. Your answers will be graded for correctness, completeness, and clarity. Points will be subtracted for incorrect arguments, arguments based on incorrect assumptions, and arguments not pertinent to the questions.

(1) Define the Dirac measure δ_{x_0} on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, for some $x_0 \in \mathbb{R}$, show that it is indeed a measure, and calculate the integral of any bounded $\mathcal{B}_{\mathbb{R}}$-measurable function with respect to it.

(2) Show that a measure μ on $\mathcal{B}_{\mathbb{R}}$ that is translation invariant and finite on all bounded elements of $\mathcal{B}_{\mathbb{R}}$ may have no atoms.

(3) Show that the sum, product, max, and min of finitely many measurable functions is measurable.

(4) For any f measurable and positive provide an explicit sequence of simple positive functions ϕ_n increasing to f from below, uniformly so when f is bounded.

(5) For μ measure on some $(X, \mathcal{B})$, show that if $f : X \to \mathbb{R}$ is in $L^1(\mu)$ then $\{x \in X : f(x) > 0\}$ is a σ-finite subset of X.

(6) Provide a proof of the Dominated Convergence Theorem by applying Fatou's Lemma twice.

(7) Give an example of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in L^1 of some measure, with $f_n \to f$ in L^1 but $f_n(x)$ not converging to $f(x)$ for any x. Then indicate a specific subsequence of your example that converges to f almost everywhere.

(8) Show that any $f : \mathbb{R} \to \mathbb{R}$ of bounded variation is the difference of two bounded, increasing functions.

(9) If $f : \mathbb{R} \to \mathbb{R}$ is monotone, show that f is measurable.

(10) Show that the weak topology on a normed space $(X, \|\,.\,\|)$ is weaker than the norm topology.

(11) Using projections and the Open Mapping theorem, prove the Closed Graph theorem (i.e. show that a closed linear map between Banach spaces is bounded).