Analysis Qualifying Exam - August 2021

Work through all parts included in the 6 items below. Start each item on a new sheet of paper. Your work will be graded for correctness, completeness, and clarity.

1. a) For a given set X and a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$, define the concept of outer measure generated by a function $\rho : \mathcal{F} \to [0, \infty]$. b) Let $X = \mathbb{R}$, and

$$\mathcal{F} = \Big\{ \emptyset, \mathbb{R}, (m - 2^{-k}, m + 1 + 2^{-k}),\ m \in \mathbb{Z},\ k \in \mathbb{N} \Big\}$$

and let $\rho : \mathcal{F} \to [0, \infty]$ be defined as

$$\rho(\emptyset) = 0, \quad \rho(\mathbb{R}) = 3, \quad \rho\big((m - 2^{-k}, m + 1 + 2^{-k})\big) = 1 + 2^{-k}.$$

If ρ^* denotes the outer measure generated by ρ, compute $\rho^*\big((0,1)\big),\ \rho^*(\{0\}),\ \rho^*\big((\frac{1}{2}, 2)\big),\ \rho^*\big((0,10)\big)$, justifying your answers.

2. a) If μ^* is an outer measure on a set X define μ^*-measurability. b) State the Hahn-Kolmogorov-Caratheodory extension theorem. c) Outline briefly how to use the theorem in b) to construct the Lebesgue measure.

3. a) State the Lebesgue dominated convergence theorem. b) Using the Lebesgue dominated convergence theorem prove that if $E = \big\{(x, y) \in \mathbb{R}^2 : x \geq 1\big\}$ then

$$\lim_{n \to +\infty} \int_E \frac{\sin\left(\dfrac{n}{x^2 + y^2}\right)}{1 + nx} dx dy = 0.$$

4. Consider the Banach spaces $(C[0,1], \|\cdot\|_\infty)$ and $(c_0, \|\cdot\|_{\ell^\infty})$ (where c_0 is the set of real valued sequences converging to 0). Let $\{x_k,\ k \in \mathbb{N}\}$ be a countable dense subset of $[0,1]$ and let $T : C[0,1] \to c_0$ be defined as $T(f) = \left\{\dfrac{f(x_k)}{k}\right\}_{k=1}^{\infty}$. a) Show that T is well-defined, linear and one-to-one. b) Show that T is bounded and compute $\|T\|$. c) Show that the inverse of T from $T(C[0,1])$ to $C[0,1]$ is not bounded. [Hint: show that there is no $A > 0$ such that $\|T(f)\|_\infty \geq A\|f\|_\infty$ for all $f \in C[0,1]$.] d) Is the range of T closed in c_0? Explain.

5. a) Define what it means for a set to be meager (or of 1^{st} category). b) Let $I = [0,1]$ be endowed with the Lebesgue measure. Prove that if $p > 1$ then $L^p(I) \subseteq L^1(I)$. c) Prove that if $p > 1$ then $L^p(I)$ is meager in $L^1(I)$.
[Hint: consider $B_n = \big\{f \in L^1(I) : \int_I |f|^p dm \leq n\big\}$ for $n \in \mathbb{N}$.]

6. a) Define orthonormal set on a Hilbert space H. b) State and prove Bessel's inequality. c) Define what it means for an orthonomal set $\{u_\alpha\}_{\alpha \in A}$ to be complete. d) State a theorem that characterizes completeness of o.n. sets in terms of Parseval's identity and Fourier series expansion.