Analysis Qualifying Exam - August 2018

- *Work through all parts included in the 8 items below.*
- *Start each item on a clean sheet of paper.*
- *Write only on one side of each sheet.*
- *Put your name on each sheet of paper you hand in.*
- *Your work will be graded for correctness, completeness, and clarity.*

1. Outline the construction of a set in $[0,1]$ which is not Lebesgue measurable.

2. Define the concept of measurability of real-valued functions on a measure space $(X,\mathcal{M})$. Show that χ_E is measurable if and only if $E \in \mathcal{M}$. Show that the sum of two measurable functions is measurable.

3. State Fatou's Lemma, on a measure space $(X,\mathcal{M},\mu)$. For given measurable sets $A, B \in \mathcal{M}$ consider the sequence of functions
$$f_n = \begin{cases} \chi_A & \text{if } n \text{ odd} \\ \chi_B & \text{if } n \text{ even.} \end{cases}$$
Give a necessary and sufficient condition on A, B so that equality holds in Fatou's lemma applied to the sequence $\{f_n\}$.

4. Make use of the Dominated Convergence Theorem, and not any other method, to evaluate the following (dm =Lebesgue measure):
$$a)\ \lim_{n\to+\infty}\int_{[0,\infty)} \frac{n(1-e^{-x/n})}{1+x^4}\,dm, \qquad b)\ \lim_{n\to+\infty}\int_{[0,\infty)} \frac{\sin(nx)}{nx^{3/2}+x^2}\,dm.$$

5. Define convergence in measure on a measure space $(X,\mathcal{M},\mu)$. Assume the following result: if $\{f_n\}$ is Cauchy in measure then there is a subsequence $\{f_{n_k}\}$ converging a.e. to a measurable function f. Using this result, and not any other method, prove: a) if $f_n \to f$ in L^1 then there exists $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e.; b) L^1 is complete.

6. Define positive/negative/null sets for a signed measure ν on $(X,\mathcal{M})$. State and prove the Hahn decomposition theorem (you can assume without proof that a measurable set with positive and finite measure contains a positive subset, with positive measure).

7. Consider the space $\mathcal{X}_0 = C[0,2] \subseteq \mathcal{X} := L^1[0,2]$, both endowed with L^1 norm w.r. to the Lebesgue measure, and the linear functional $T_0 : \mathcal{X}_0 \to \mathbb{C}$ defined as
$$T_0 f = \int_0^2 te^{-t} f(t)\,dt,$$
   a) Show that T_0 is bounded, and compute $\|T_0\|$ from the definition of operator norm (and not any other method).
   b) Show that if $f_0 = \chi_{[0,1]}$, an element of $\mathcal{X} \setminus \mathcal{X}_0$, then there is a linear extension T of T_0 defined on the space $\mathcal{X}_0 + \mathbb{R}f_0$ which is not bounded (find such T explicitly).

8. State and prove the Schwarz inequality and the Triangle inequality on a Hilbert space H.