ANALYSIS QUALIFYING EXAM AUGUST 2017

Instructions:

- Do ten questions.
- Do at most one question on each sheet of paper.
- Put your name on each sheet of paper you hand in.
- Use a paper clip to put your answers together.

(1) Define the Lebesgue outer measure $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$. Show $m^*(I) = \ell(I)$ for any interval I, where $\ell(I)$ denotes the length of I. Remember to include the cases where I is closed, open, half-open, and unbounded.

(2) Give Carathéodory's definition for a subset of $\mathbb{R}$ to be Lebesgue measurable. Show that if A and B are measurable, so is $A \cup B$. Show that if A and B are measurable and $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

(3) Use the axiom of choice to show the existence of a non-measurable set.

(4) Define what it means for a function $f : \mathbb{R} \to [-\infty, \infty]$ to be measurable. Show that if f is measurable, and B is Borel, then $f^{-1}(B)$ is measurable.

(5) Suppose that we have defined $\int f$ for f simple and supported on a set of finite measure, and that we know that this definition is additive. Show how to define $\int f$ for any measurable $f \geq 0$. Show that $\int (f + g) = \int f + \int g$ whenever $f, g \geq 0$.

(6) Give the statement of Vitali's Convergence Theorem for uniformly integrable sequences of functions. Define what it means for a function to be absolutely continuous. Show that if f is absolutely continuous and increasing on $[a, b]$, then $\int_a^b f' = f(b) - f(a)$.

(7) Define the spaces $L^p(E)$ for $1 \leq p < \infty$ where E is a measurable subset of $\mathbb{R}$. State and prove Minkowski's inequality. You may quote Hölder's inequality without proof.

(8) Show that the simple functions are dense in $L^p(E)$ for $1 \leq p < \infty$ for any measurable E. Show that the step functions are dense in $L^p(I)$ for $1 \leq p < \infty$ for any interval bounded I.

(9) Consider the sequence of functions $f_n(x) = \sin(nx)$ on $[0, 1]$. Show that f_n converges weakly to the zero function on $L^p([0, 1])$. (Hint, first show that $\int_{[0,1]} f_n I_{[a,b]} \to 0$ for any $0 \leq a \leq b \leq 1$.)

(10) State the Tonelli and Fubini Theorems. You may assume the construction of the product measure without proof. Give a counterexample to Tonelli's Theorem that shows the necessity of the assumptions that the measure spaces are σ-finite.

(11) Define what is meant by a signed measure on a sigma-algebra. Define what it means for a measurable set to be positive with respect to a signed measure. Show that for any signed measure, then there exists a positive measurable set, whose complement is also positive. (For simplicity, you may assume throughout that the signed measure is bounded.)

Date: 08/14/17.

(12) State and prove the Radon-Nikodyn Theorem. You may assume that all measures are finite and unsigned. You may assume the Riesz Representation Theorem for Hilbert spaces, or the Jordon/Hahn Decomposition Theorem for signed measures, without proof.

(13) State the Closed Graph Theorem. Show that a closed subspace Y of a Banach space X has a closed complement if and only if there is a continuous projection $X \to Y$.