May 2021 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 5 questions with 4 parts. Each part is worth 5 points for a total of 100 points. There are three pages including this one.

In what follows all rings R are assumed to have a multiplicative identity 1_R. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example.

1. Groups.

a. What is the maximum order of an element in the symmetric group S_7?

b. Prove that if $|G|$ is odd then there are no non-trivial homomorphisms $G \to \{\pm 1\}$.

c. Let p be prime. A group of order p^n acts on a finite set S with fixed set S_0. Prove that $|S| \equiv |S_0| \mod p$.

d. Let p be a prime. Prove that if $p||G|$ then G contains an element of order p.

2. Rings.

a. Prove that if R is a ring and $\mathfrak{p}$ is a maximal ideal then $R_{\mathfrak{p}}$ has exactly one maximal ideal.

b. Let $R = \mathbb{Z}/24\mathbb{Z}$ and $\mathfrak{p} = (\overline{3})$ explicitly determine $R_{\mathfrak{p}}$.

c. Prove that $\mathbb{Q}[x]$ is a principal ideal domain.

d. Prove that if R is an integral domain then a polynomial $f \in R[x]$ of degree $n > 0$ has at most n distinct roots.

3. Fields.

a. Prove that any finite extension of fields $F \supset K$ is algebraic.

b. (i) Define what it means for a field extension $F \supset K$ to be *separable*.
(ii) Define what it means for $F \supset K$ to be *Galois*.

c. Prove that if $F \supset K$ is a finite separable extension, then there are only finitely many intermediate field extensions $F \supset L \supset K$.

d. Prove that if K is an infinite field of characteristic p and $F = K(u, v)$ is an extension of degree p^2 such that $u^p, v^p \in K$ then there are an infinite number of intermediate fields $F \supset L \supset K$.

4. Modules.

a. Give an example of an exact sequence of R-modules $0 \to A \to B \to C \to 0$ which is not split. Make sure you explictly say what ring R you are using and why the sequence is not split.

b. Prove that if $0 \to A \to B \to P \to 0$ is a short exact sequence of R modules and P is projective, then $B \simeq A \oplus P$.

c. Prove that if R is a non-zero commutative ring such that every submodule of a free module is free then R is a principal ideal domain.

d. Let R be a domain and let A be a non-zero R-module. Prove or give a counter example to the following statement.

The dual module A^* is non-zero.

5. Linear Algebra.

a. Give an explicit example of a non-zero alternating bilinear form $(\mathbb{R}^3)^2 \to \mathbb{R}$.

b. Find the possible Jordan canonical forms for a matrix whose characteristic polynomial is $(x-1)(x^3-1)$.

c. Prove that if A, B are $n \times n$ complex matrices such that $AB = BA$ then A and B have a common eigenvector.

d. Let $V = \mathbb{R}^2$ with $\mathbb{R}[x]$ module structure given by the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a vector by $\pi/2$. Prove that V is a simple $\mathbb{R}[x]$-module and determine the minimial polynomial of T.