May 2016 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty.

In what follows, if n is a positive integer, the ring of integers modulo n is denoted by $\mathbb{Z}_n$; if m is an arbitrary integer, its class modulo n is denoted by $\overline{m} \in \mathbb{Z}_n$. For a group G, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G, under composition of maps. When you are doing a problem with multiple parts, you can use the earlier part to prove the later ones even if you do not prove the earlier part.

1. Prove or give a counterexample.

a. (5 points) All groups of order 35 are isomorphic to the cyclic group $\mathbb{Z}_{35}$.

b. (5 points) The polynomial $q(x) = 3x^4 + 15x^3 + 25x^2 + 45$ is irreducible over $\mathbb{Q}$.

c. (5 points) A finite abelian group G is cyclic if and only if its group of automorphisms Aut(G) is abelian.

2. (10 points) State completely any theorem that you are quoting.

Suppose that F is a field. Let F^* denote the multiplicative group of units in F.

a. Prove that all finite subgroups of F^* must be cyclic.

b. Do you need the fact it is finite in a? Does the result of 2a hold if the subgroup is not finite? Justify your answers.

3. (15 points) Let $f \in \mathbb{Q}[x]$ be a polynomial of degree n such that the Galois group G of the splitting field of f over $\mathbb{Q}$ is isomorphic to S_n, the symmetric group.

a. Show that f must be irreducible.

b. Let $n > 2$. Show that if u is a root of f, then the identity is the only automorphism of $\mathbb{Q}[u]$.

c. Give an example of a polynomial f of degree n whose Galois group is S_n. Justify your answer.

4. (3+4+3 points)

a. Define a prime ideal in a ring R.

b. Prove that if R is a commutative ring then it must have at least one prime ideal.

c. Give an example of a ring R with a prime ideal that is not a maximal ideal. Justify your example.

5. Let R be a commutative ring and M be a finitely generated R module.

a. (2 points) Define a *free R-module.*

b. (3 points) Show that if M is a finitely generated R-module then there is a free module F of finite rank and an epimorphism (surjective homomorphism) of R-modules $\phi\colon F \to M$.

c. (7 points) If M and N are two finitely generated R-modules, show that $\mathrm{Hom}_R(M, N)$, the set of all R module homomorphisms from $M \to N$, is an R-module and that it is finitely generated.

6. (4 points each) Let $S = M(n, \mathbb{R})$ denote the ring of $n \times n$ matrices over the real numbers. Let $\mathrm{GL}(n, \mathbb{R})$ be the group of invertible matrices in S. Let $P \in \mathrm{GL}(n, \mathbb{R})$. We say that P diagonalizes a matrix A if $P^{-1}AP$ is a diagonal matrix.

a. Prove that the set of all matrices $S(P)$ diagonalized by P is a subring of S and that $S(P)$ is a commutative ring.

b. Prove that for any two invertible matrices P and Q, $S(P)$ and $S(Q)$ are isomorphic.

c. Prove that $\cap_{P \in \mathrm{GL}(n,\mathbb{R})} S(P)$ is isomorphic to $\mathbb{R}$.

7. a. (6 points) Suppose that V and W are vector spaces of finite dimension m and n respectively and that $L : V \to W$ is a linear map and the kernel of L has dimension r.

Prove that there are bases for V and W such that the matrix of L in those bases has the form $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ where A is an $(m - r) \times (m - r)$ matrix.

b. (6 points) Let P_4 be the vector space of all polynomials of degree less than 4 over a field K.

$<, >$ is a bilinear form on P_4 defined by $< f, g >= f(1)g(1) + f(-1)g(-1) + f(2)g(2) + f(-2)g(-2)$.

Determine all the fields K, if any, over which the bilinear form $<, >$ is non degenerate and the fields K, if any, over which it is degenerate. Prove your answer.

You do not need to "list" all the fields, if your list is infinite. You can give necessary and sufficient conditions that amounts to determining the fields up to isomorphism. (You do not need to show that the form is bilinear).

c. (7 points) $V = \mathbb{R}^4$ is a vector space with a scalar product $*$ given by $(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4) = 4(x_1y_2 + x_2y_1) + 2(x_1y_4 + x_4y_1) - 2(x_2y_2 + x_3y_3) + x_2y_4 + x_4y_2 + x_3y_4 + x_4y_3 + x_4y_4$.

Find an orthogonal basis for $\mathbb{R}^4$ in this scalar product.

8. (7 points) Let K be a field of chracteristic zero. tr denotes the trace. Prove that for a square matrix A over K, if $tr(A^k) = 0$ for all positive integers k, then A must be nilpotent. What about the converse?