Qualifying Examination
August 2019

- You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part.

- $\mathbb{Z}$ denotes the group or ring of integers with the usual operations.

- $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the groups or fields of rational, real, and complex numbers, respectively, with the usual operations.

- By a PID, we mean a Principal Ideal Domain.

- If $n \geq 3$ is an integer, we denote by D_{2n} the dihedral group with $2n$ elements.

- For a square-free integer D other than 1, define $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$.
- Given a finite dimensional vector space V over a field F and an F-linear operator $T : V \to V$, we endow V with the structure of an $F[X]$-module such that $X \cdot v = T(v)$ for all $v \in V$.

Algebra Qualifying Exam

(I) Groups

(1a) [3 points] Show that every finitely generated subgroup of the additive group $(\mathbb{Q}, +)$ is cyclic, and that $(\mathbb{Q}, +)$ is not finitely generated.

(1b) [3 points] Determine if the following statement is true or false and *substantiate* your answer.

> Let G be an infinite group such that for each positive integer n, G has only finitely many subgroups of index n. Then G is a finitely generated group.

(2) [8 points] Show that a group of order 48 is not simple.

(3) [6 points] Let p, q be primes, $p > q > 2$. Let G be a group of order pq^2. Show that G has a subgroup of order pq. [*Hint*: First show that G has a unique p-Sylow subgroup.]

(II) Rings and Fields

(4) Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a squarefree integer greater than 3.

(a) [4 points] Prove that 2, $\sqrt{-n}$ are irreducible in R.

(b) [2 points] Prove that R is not a UFD. [For this part, feel free to use (without proof) the fact that $1 + \sqrt{-n}$ is also irreducible in R.]

(5) Let R be an integral domain such that every prime ideal in R is principal. Show that R is a PID by going through the following steps.

(a) [2 points] Assume for a contradiction that the set of ideals of R which are not principal is non-empty and prove that this set has a maximal element under inclusion (which by hypothesis is not prime).

(b) [1 point] Let I be an ideal which is maximal with respect to being non-principal and let $a, b \in R$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Consider the ideals $I_a := (I, a)$, $I_b := (I, b)$, and $J := \{r \in R \mid rI_a \subseteq I\}$. Show that I_a and J are principal ideals and let α and β be generators of I_a and J, respectively.

(c) [3 points] Finally, show that $I = (\alpha\beta)$, providing thus a contradiction. This shows that R has to be a PID.

(6) [5 points] Determine if the following statement is true or false and *substantiate* your answer.

> A field extension is finite if and only if it is finitely generated by finitely many algebraic elements.

(7)[8 points] Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and $F = \mathbb{Q}(\sqrt{2})$. Show that K/F is Galois and $Gal(K/F) \simeq D_8$.

(III) Modules and Linear Algebra

(8) [3 points] Determine if the following statement is true or false and *substantiate* your answer.

> Over an algebraically closed field, two $n \times n$ matrices A and B are similar if they have the same characteristic and minimal polynomials.

(9a) [2 points] Let A be a 5×5 complex matrix with characteristic polynomial $c(X) = (X-2)^3(X+7)^2$ and minimal polynomial $m(X) = (X-2)^2(X+7)$. Find the Jordan canonical form of A.

(9b) [3 points] Let V be a real vector space of dimension ≥ 3 and $T : V \to V$ a linear map. Show that there exists a non-zero subspace W of V, $W \neq V$, such that $T(W) \subseteq W$.

(9c) [4 points] Let $f(X) \in \mathbb{C}[X]$ be a polynomial of positive degree. Prove that all $n \times n$ matrices with characteristic polynomial $f(X)$ are similar if and only if $f(X)$ has no repeated factors in its unique factorization in $\mathbb{C}[X]$.

(10) Let V be a finite dimensional vector space over a field K. Let $T : V \to V$ be a linear transformation and $W \leq V$ a subspace of V that is T-invariant, i.e. $T(W) \subseteq W$. Let $m(X)$, $m_1(X)$, and $m_2(X)$ be the minimal polynomial of T as a linear operator on V, W, and V/W, respectively.

(a) [3 points] Prove that $m(X)$ divides $m_1(X) \cdot m_2(X)$.

(b) [3 points] Prove that if $m_1(X)$ and $m_2(X)$ are relatively prime, then

$$m(X) = m_1(X) \cdot m_2(X).$$

(c) [2 points] Give an example of a case in which $m(X) \neq m_1(X) \cdot m_2(X)$.