August 2021 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 5 questions with 4 parts. Each part is worth 5 points for a total of 100 points. There are three pages including this one.

In what follows all rings R are assumed to have a multiplicative identity 1_R. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example.

1. Groups.

a. Prove or give a counter-example. If every proper subgroup of a group is abelian then the group is abelian.

b. Prove that if G is a subgroup of S_n such that $G \cap A_n = \{e\}$ then $|G| = 2$.

c. Prove that every finite p-group has a non-trivial center.

d. Give an example of a solvable finite group which is not nilpotent.

2. Rings. In this part all rings are assumed to be commutative.

a. Prove or give a counter-example. If R is a domain and $f \in R$ is irreducible then (f) is a prime ideal.

b. Let R be a ring. Prove that an element $r \in R$ is nilpotent if and only if $S^{-1}R = 0$ where $S = \{r^n | n \geq 0\}$.

c. Prove that the set of nilpotent elements in a ring R is the intersection of all prime ideals. [Hint: Use part (b)]

d. Prove that $\mathbb{Q}[x]$ is a principal ideal domain.

3. Fields.

a. Prove or give a counter-example. If $K \subset F$ is algebraic then $[F : K] < \infty$.

b. (i) Define what it means for a field extension $K \subset F$ to be *separable.*

(ii) Define what it means for $K \subset F$ to be *Galois.*

c. Let F be the splitting field of a separable polynomial $f \in K[x]$ of degree n. Show that $[F : K] | n!$. (Note that the the assumption that f is separable is not necessary, but makes the proof easier.)

d. What is the degree of the extension $\mathbb{Q} \subset \mathbb{Q}(i, e^{2\pi i/3})$? Find a primitive element for this extension.

4. Modules.

a. Prove or give a counter-example. Every module is a submodule of a projective module.

b. Give an example of a flat module which is not projective.

c. Let p be prime and let $H = \{x \in \mathbb{Q}/\mathbb{Z} | \exists n \in \mathbb{N}, p^n x = 0\}$. Prove that H is a divisible group.

d. Let R be a commutative ring. Prove or give a counter-example. If $A \to B$ is a surjective morphism of R-modules then $\mathrm{Hom}_R(B, D) \to \mathrm{Hom}_R(A, D)$ is a monomorphism for all R-modules D.

5. Linear Algebra.

a. Let R be a commutative ring. Prove that if $m < n$ then there is a non-zero alternating R-linear form $(R^n)^m \to R$.

b. Find the possible rational canonical forms for a matrix (with rational entries) whose characteristic polynomial is $(x-1)(x^3-1)$.

c. Prove that if $T\colon V \to V$ is a linear transformation then V is T-cyclic if and only if the minimal polynomial of T equals the characteristic polynomial.

d. Find a decomposition of the group ring $R = \mathbb{C}[\mathbb{Z}_3]$ into simple R-modules. (Recall the the group ring $\mathbb{C}[G]$ of a finite group G is the set of all formal sums $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{C}$. Addition is componentwise and multiplication is determined by the rule that $g \cdot g' = gg'$.)