August 2016 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 9 questions with multiple parts. There are three pages including this one.

In what follows, if n is a positive integer, the ring of integers modulo n is denoted by $\mathbb{Z}_n$; if m is an arbitrary integer, its class modulo n is denoted by $\overline{m} \in \mathbb{Z}_n$. For a group G, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G, under composition of maps. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part.

1. The exponent of a group G is the smallest positive integer r, if it exists, such that $g^r = e$ for all $g \in G$. If no such r exists, the exponent of G is said to be infinite.
Prove or give a counterexample.

a. (4 points) The exponent of a finite group divides the order of the group.

b. (4 points) A finite group G is cyclic if and only if the exponent of G equals the order of G.

2. a. (6 points) Prove that a group of order 48 is not simple.

b. (4 points) Prove that if H is the only subgroup of order n in a group G, then H must be normal in G.

c. (6 points) If a group G has a normal subgroup H of order 3 which is not contained in the center of G, prove that G has a subgroup of index 2.

3. Let E be an extension field of a field F. An element $s \in E$ is algebraic over F if there is a nonzero polynomial $f \in F[x]$ such that $f(s) = 0$. In a and b below, use the above definition of an algebraic element.

a. (7 points) Prove that if s and t in E are algebraic over F then so are $s + t$ and st.

b. (8 points) Let $\hat{\mathbb{Q}} = \{s \in \mathbb{C} \,|\, s \text{ is algebraic over } \mathbb{Q}\}$. Show that $\hat{\mathbb{Q}}$ is a field extension of $\mathbb{Q}$ and the Galois group $G(\hat{\mathbb{Q}}/\mathbb{Q})$ is an infinite group.

4. a. (4 points) Let K be a field. Prove that a polynomial $f \in K[x]$ of positive degree n can have at most n roots in K.

b. (4 points) Give an example of a ring R and a polynomial $f \in R[x]$ of positive degree n (for some specific n) which has more than n roots in R. Justify your example.

5. a. (3 points) Define a prime ideal in a ring R.

b. (8 points) Let R be a commutative ring and $x \in R$ be neither a unit nor a nilpotent element. (That is, x is not a unit and $x^n \neq 0$ for any positive integer n.) Show that there is a prime ideal P of R containing x and a prime ideal Q of R that does not contain x.

6. Let R be a commutative ring.

a. (8 points) Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence of R-modules. Prove that i implies ii.

i. There is a homomorphism $\phi : C \rightarrow B$ such that $g \circ \phi = id$

ii. B is isomorphic to $A \bigoplus C$.

b. (5 points) If M and N are two finitely generated R-modules, show that $\mathrm{Hom}_R(M, N)$, the set of all R module homomorphisms from $M \rightarrow N$, is an R-module and that it is finitely generated.

7. Let k be a field. An $n \times n$ matrix A over k is diagonalizable if there is an $n \times n$ matrix P over k such that $P^{-1}AP$ is diagonal. We then say P diagonalizes A or that A is diagonalized by P. Let D_P be the set of all matrices diagonalized by P.

a. (4 points) Show that D_P is a subring of the ring $M(n, k)$ of all $n \times n$ matrices over k.

b. (6 points) Show that D_P is a commutative ring and show that D_P is not simple. (A ring is simple if it has no two sided ideals other than itself and the trivial one 0).

8. a. (6 points) Let P_4 be the vector space of all polynomials of degree less than 4 over a field K.

$<,>$ is a bilinear form on P_4 defined by

$$< f, g >= f(1)g(1) + f(-1)g(-1) + f(2)g(2) + f(-2)g(-2).$$

Determine all the fields K, if any, over which the bilinear form $<,>$ is non degenerate and the fields K, if any, over which it is degenerate. Prove your answer.

You do not need to "list" all the fields, if your list is infinite. You can give necessary and sufficient conditions that amount to determining the fields up to isomorphism. (You do not need to show that the form is bilinear).

b. (6 points) $V = \mathbb{R}^4$ is a vector space with a scalar product $*$ given by $(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4) = 4(x_1y_2 + x_2y_1) + 2(x_1y_4 + x_4y_1) - 2(x_2y_2 + x_3y_3) + x_2y_4 + x_4y_2 + x_3y_4 + x_4y_3 + x_4y_4$.

Find an orthogonal basis for $\mathbb{R}^4$ in this scalar product.

9. Let $n \geq 2$ and A be an $n \times n$ singular matrix of rank $n - 1$ over a field k.

a. (4 points) Show that the adjoint of A is diagonalizable.

b (3 points) Determine characteristic polynomial of the adjoint of A?