MAY 2019 ANALYSIS QUALIFYING EXAM

Instructions: Do all 8 problems. Use a separate sheet for each problem.

1. For x, t real let $\varphi(t,x) = t\sin(x) + \cos(x)$.
(a) Prove that

$$\int_{1/n}^{n} \frac{\sin x}{x}\,dx = \int_0^\infty \frac{e^{-t/n}\varphi(t,1/n) - e^{-nt}\varphi(t,n)}{t^2+1}dt.$$

Hint: Use that $1/x = \int_0^\infty e^{-xt}dt$ for $x > 0$ on the left hand side.
(b) Find the limit of the expression in part (a) as $n \to \infty$.
Justify your steps in parts (a) and (b).

2. (a) Let $\{B_n\}_{n=1}^\infty$ be a decreasing sequence of measurable subsets of a measure space $(X, \mathcal{S}, \mu)$, in the sense that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$. Let $B = \bigcap_{n=1}^\infty B_n$. Is it true that

$$\lim_{n\to\infty} \mu(B_n) = \mu(B)?$$

Prove this fact or provide a counterexample.
(b) Suppose that $\mu(X) < \infty$. Assume that $C_n \in \mathcal{S}$, $n = 1, 2, \ldots,$ satisfy $\mu(C_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Prove that there exists a set $S_0 \in \mathcal{S}$ such that $\mu(S_0) > 0$ and that every $x \in S_0$ lies in C_j for infinitely many $j \in \mathbb{N}$. (*Hint*: Put $B_n = \bigcup_{k=n}^\infty C_k$.)

3. Let λ be Lebesgue measure on the σ-algebra $\mathcal{L}$ of all Lebegue measurable subsets of $[-1,1]$. Define a measure ν on $([-1,1], \mathcal{L})$ by setting

$$\nu = \delta_0 + h\,d\lambda,$$

where δ_0 is Dirac mass at zero and $h(t) = e^t$, $t \in [-1,1]$. Which of the two statements $\nu \ll \lambda$, $\lambda \ll \nu$ is true and which one is false? Justify your answers in both statements. For the true statement find the associated Radon-Nikodym derivative.

4. (a) State Tonelli's theorem and Fubini's theorem.
(b) Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$, let

$$g(x) = \int_x^1 t^{-1} f(t)dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

5. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. For a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on $\mathcal{H}$ satisfying:

(α) $\langle Af, f\rangle \geq 0 \qquad (f \in \mathcal{H})$
(β) $\langle Af, g\rangle = \langle f, Ag\rangle \qquad (f, g \in \mathcal{H})$

we define $tr(A^2)$, the trace of A^2, by

$$tr(A^2) = \sum_{n=1}^{\infty} \langle A^2 e_n, e_n\rangle \ \left(= \sum_{n=1}^{\infty} \langle Ae_n, Ae_n\rangle \right).$$

Prove that the definition of $tr(A^2)$ above is not dependent on the orthonormal basis, i.e. prove that if $\{d_m : m \in \mathbb{N}\}$ is another orthonormal basis on $\mathcal{H}$, then

$$\sum_{n=1}^{\infty} \langle A^2 e_n, e_n\rangle = \sum_{m=1}^{\infty} \langle A^2 d_m, d_m\rangle.$$

Hint: Expand Ae_n in terms of the basis $\{d_m\}$.

6. (a) State the Lebesgue differentiation theorem.
(b) Prove that there does not exist a Lebesgue measurable subset E of $[0, 1]$ such that the Lebesgue measure of $E \cap [x, 1]$ is equal to $(1 - x)^2$ for all $x \in [0, 1]$.

7. Let f, g be elements of an inner product space V over $\mathbb{R}$. Prove that $\langle f, g\rangle = 0$ if and only if for all $\lambda \in \mathbb{R}$

$$\|f\| \leq \|f + \lambda g\|.$$

8. (a) State (but do not prove) Fatou's lemma.
(b) Suppose that f_n is a sequence of measurable functions on a measure space that satisfy $f_n \leq g$ for some $g \in L^1$. Using part (a) show that

$$\limsup_{n\to\infty} \int f_n \, d\mu \leq \int \limsup_{n\to\infty} f_n \, d\mu.$$

(c) For the sequence $f_n(x) = (-1)^n \chi_{[0,1]} + (-1)^{n+1} \chi_{(1,3]}$ defined on $[0, 3]$, show that

$$\int_0^3 \limsup_{n\to\infty} f_n dx - \limsup_{n\to\infty} \int_0^3 f_n dx = \liminf_{n\to\infty} \int_0^3 f_n dx - \int_0^3 \liminf_{n\to\infty} f_n dx$$