AUGUST 2019 ANALYSIS QUALIFYING EXAM

Instructions: Do all 8 problems. Use a separate sheet for each problem.

1. Let

$$f(x,y) = \begin{cases} x^{-4} & \text{if} \quad y \le x^2,\ x \neq 0 \\ -y^{-2} & \text{if} \quad y > x^2 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Apply Tonelli's Theorem to prove that f is not integrable on $A = [0,\infty)^2$, but it is integrable on $B = [0,\infty) \times [1,\infty)$. Apply Fubini's Theorem to compute the integral

$$\int\int_B f(x,y)dxdy.$$

2. State the dominated convergence theorem. Let

$$f_n(x) = \frac{\sin(nx^{1/3})}{x(n + x^{1/3})}, \qquad x > 0$$

Show that $f_n \in L^1([0,\infty))$ and that

$$\lim_{n\to+\infty} \int_0^\infty f_n(x)dx = 0.$$

3. Let A be a measurable subset of $[0,1]$ of positive measure. Show that there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ such that $x_1 - x_2$ is a rational number.

4. Let f be a measurable function on a measure space (X,μ).
(a) Prove that if f is integrable then the series

$$\sum_{n=1}^{\infty} \mu(\{x \in X : \ |f(x)| > n^2\})$$

converges. Provide an example to show that the converse implication is false.
(b) Show that if $\mu(X) < \infty$ and

$$\sum_{n=1}^{\infty} n^2 \mu(\{x \in X : \ |f(x)| > n^2\})$$

converges, then f must be integrable over X.

5. Let T be a linear operator from a Hilbert space V (over $\mathbb{C}$) to V that satisfies

$$\|f\| = \frac{1}{5}\|T(f)\|$$

for all $f \in V$. Prove that for all $f, g \in V$ we have

$$\langle T(f), T(g)\rangle = 25\langle f, g\rangle.$$

6. Consider the space $\mathcal{C}[-1,1]$ of continuous functions on $[-1,1]$, equipped with the L^1 norm with respect to the Lebesgue measure, and the linear functional $T_0 : \mathcal{C}[-1,1] \to \mathbb{R}$ defined by

$$T_0(f) = \int_{-1}^{1} |t| f(t)dt.$$

(a) Show that T_0 is bounded, and compute $\|T_0\|$.
(b) Let $f_0 = \chi_{[0,1]}$ and consider the operator $T : \mathcal{C}[-1,1] + \mathbb{R}f_0 \to \mathbb{R}$ defined as

$$T(f + \lambda f_0) = \int_{-1}^{1} |t| f(t)dt + \lambda, \qquad f \in \mathcal{C}[-1,1], \lambda \in \mathbb{R}.$$

(i) Show that T is a linear functional that extends T_0.
(ii) Show that T is unbounded on $\mathcal{C}[-1,1] + \mathbb{R}f_0$, equipped with the L^1 norm.

7. Let $g(t)$ be a nonnegative measurable function on the real line. Show that

$$\left(\int_1^\infty g(t)\, dt\right)^2 \le 3\int_1^\infty t^{4/3} g(t)^2\, dt.$$

Assuming that the right-hand side is finite, find all functions g for which equality holds in the preceding inequality.

8. Let f be a complex-valued measurable function on a measure space (X, μ) that satisfies:

$$\int_{|f|<n} \frac{|f|^2}{\frac{1}{n} + |f|}\, d\mu \le 1$$

for all $n = 1, 2, \ldots$. Find the limit of

$$\int_{|f|\ge n} \frac{|f|^2}{\frac{1}{n} + |f|}\, d\mu$$

as $n \to \infty$.