Analysis Qualifying Exam, August 2020

Instructions: *Do all 8 problems. Use a separate sheet for each problem. Your work will be graded for correctness, completeness, and clarity.*

1a. Let X be a set and let $\mathcal{A}$ be an algebra of subsets of X. Define the notion of a premeasure μ_0 and its induced outer measure μ_0^*. Define what it means for a subset of X to be μ_0^*-measurable.

b. Let $X = \mathbb{R}$. Give the definition of an algebra and an associated premeasure used in the construction of the Lebesgue measure m.

c. By assuming that the induced outer measure m^* from part (b) satisfies

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i), a_i < b_i, a_i, b_i \in \mathbb{R} \right\} \quad (A \subseteq \mathbb{R}),$$

show that for any set $A \subseteq \mathbb{R}$, there is a G_δ set $G \supseteq A$ with $m(G) = m^*(A)$.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |f_n|^2 dm \leq 1$ for all n. Assume that there exists a Lebesgue measurable function f such that

$$\int_0^1 |f_n - f| dm \to 0 \quad \text{as } n \to \infty.$$

a. Show that $\int_0^1 |f|^2 dm \leq 1$.

b. Does it follow that

$$\int_0^1 |f_n - f|^2 dm \to 0 \quad \text{as } n \to \infty?$$

3a. Let $M > 0$ and suppose that $f : \mathbb{R} \to \mathbb{R}$ is M-Lipschitz, i.e., $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Show that if A is Lebesgue measurable, then $f(A)$ is Lebesgue measurable.

b. Let $f(x) = x^2 - 2^x$. Show that if N is a Lebesgue null set, then so is $f(N)$.

4. Let f be a Borel measurable function on $[0, \infty)$ and define

$$F(s) = \int_0^{\infty} \frac{f(x)}{(1 + sx)^2} dm(x) \quad (s > 0).$$

a. Show that if the function $x \mapsto \frac{f(x)}{x}$ is integrable on $[0, \infty)$, then F is finite a.e. and that F is integrable over $[0, \infty)$.

b. Show that if f is non-negative and F is bounded, then f must be integrable.

c. Assume that f is continuous and that the limit $a := \lim_{x \to \infty} f(x)$ exists and is finite. Find $\lim_{s \to 0} sF(s)$ and justify your answer.

5. Let X and Y be Banach spaces and let $T : X \to Y$ be an injective, bounded linear operator. Denote the range of T by $R(T)$. Show that the following are equivalent:

   a. The inverse of T on $R(T)$ is continuous.
   b. There exists a positive constant C such that $\|Tx\| \geq C\|x\|$ for all $x \in X$.
   c. $R(T)$ is a closed subset of Y.

6a. State the Closed Graph Theorem.

b. Let H be a Hilbert space and let $T : H \to H$ be a linear map (not assumed to be bounded). Suppose that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Show that T must be bounded.

7. Let X be a normed space and let X^* be its dual space; denote $(X^*)^*$ by X^{**}.

   a. Show that the map $X \ni x \mapsto \widehat{x} : X \to X^{**}$ defined by

   $$\widehat{x}(f) = f(x) \quad (x \in X, f \in X^*)$$

   is a linear isometry.

   b. Let $1 \leq p \leq 2$ and let

   $$X = \ell^p(\mathbb{N}) = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}, n \in \mathbb{N}, \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

   For which values of p is the following statement true: for any $f \in X^*$, there exists $x \in X$ with $\|x\| = 1$ such that $\|f\| = |f(x)|$? Justify your answer.

8. Let f be a non-negative Borel measurable function on $[0, 1]$. For $1 \leq p \leq \infty$, let L^p denote $L^p([0, 1], \mathcal{B}_{[0,1]}, m)$ equipped with the p-norm $\|\cdot\|_p$.

   a. Show that if $f \in L^\infty$, then for each $\varepsilon > 0$ there exists $k \leq \left\lceil \frac{\|f\|_\infty}{\varepsilon} \right\rceil$ pairwise disjoint measurable sets $A_1, \ldots, A_k$ such that the function $\phi := \sum_{j=1}^{k} a_j \chi_{A_j}$, with $a_j := \frac{1}{m(A_j)} \int_{A_j} f\, dm$, satisfies $\|f - \phi\|_1 < \varepsilon$.

   b. Show that if $f \in L^2$, then for each $t > 0$,

   $$\int_{\{f>t\}} f\, dm \leq \frac{\|f\|_2^2}{t}.$$

   c. Show that the conclusion in part (a) holds if f belongs to L^2 (instead of L^∞) but with $k \leq \left\lceil \frac{4\|f\|_2^2}{\varepsilon^2} \right\rceil$.