ANALYSIS QUALIFYING EXAM MAY 2017

Instructions:

- Do ten questions.
- Do at most one question on each sheet of paper.
- Put your name on each sheet of paper you hand in.
- Use a paper clip to put your answers together.

(1) Define the Lebesgue outer measure $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$. Give Carathéodory's definition for a subset of $\mathbb{R}$ to be Lebesgue measurable. Show that open intervals and sets of outer measure zero are Lebesgue measurable.

(2) Show that if A is a Lebesgue measurable subset of $\mathbb{R}$, then there exists a F_σ set B and a G_δ set C such $B \subseteq A \subseteq C$ and $m(C \sim B) = 0$. Conclude that the Lebesgue measurable sets are contained in the minimal σ-algebra containing open sets and sets whose outer measure is zero.

(3) Define what it means for a function $f : E \to [-\infty, \infty]$ to be measurable, where E is a measurable subset of $\mathbb{R}$. Show that if $m(E) < \infty$ and $|f(x)| \le M$ for all $x \in E$, then there exists two sequences of simple functions $\psi_n, \varphi_n : E \to [-M, M]$ such that $\psi_n \le f \le \varphi_n$, and $|\psi_n(x) - \varphi_n(x)| < 1/n$ for almost every $x \in E$.

(4) Suppose that we have defined $\int f$ for $f \ge 0$ measurable, and that we know that this definition is additive. Show how to define $\int f$ for any measurable f for which $\int |f| < \infty$. Show that $\int (f + g) = \int f + \int g$ whenever f and g are measurable with $\int |f| < \infty$ and $\int |g| < \infty$. Show also that $\left|\int f\right| \le \int |f|$.

(5) Prove that $\int_0^\infty (1 + t^2/n)^{-n}\, dt \to \int_0^\infty e^{-t^2}\, dt$ as $n \to \infty$.

(6) Define what it means for $\varphi : \mathbb{R} \to \mathbb{R}$ to be convex. Show that if φ is convex, and $x_1 \le x_2 \le x_3$, then $\dfrac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \le \dfrac{\varphi(x_3) - \varphi(x_1)}{x_3 - x_1}$. Show also that for each $x_0 \in \mathbb{R}$ there exists $m \in \mathbb{R}$, such that $\varphi(x) \ge \varphi(x_0) + m(x - x_0)$. Deduce that if $\varphi \ge 0$, then for any integrable function $f : [0, 1] \to \mathbb{R}$ we have $\varphi\left(\int_{[0,1} f\right) \le \int_{[0,1]} \varphi \circ f$.

(7) Define the spaces $L^p(E)$ for $1 \le p \le \infty$ where E is a measurable subset of $\mathbb{R}$. Show that L^p is complete for $1 \le p < \infty$.

(8) State and prove Young's inequality, including conditions for equality. (A picture proof is adequate.) State and prove Hölder's inequality for $1 < p < \infty$, $q = p/(p - 1)$ including conditions for equality.

(9) State the Tychonoff Product Theorem. Define the weak topology on a Banach space X. Define the weak* topology on the dual Banach space X^*. State and prove Alaoglu's Theorem.

(10) Show that there is a bounded sequence f_n in $L^1([0, 1])$ such that no subsequence f_{n_k} converges weakly.

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(11) Prove the following form of the Hahn-Banach Theorem. If Y is a subspace of X, and if there exists a sub-additive, positively-homogeneous function $\rho : X \to [0, \infty)$, and if $\varphi : Y \to \mathbb{R}$ is a linear function satisfying $\varphi(y) \leq \rho(y)$ for all $y \in Y$, then there is a linear function $\tilde{\varphi} : X \to \mathbb{R}$ satisfying $\tilde{\varphi}(x) \leq \rho(x)$ for all $x \in X$, and $\tilde{\varphi}|_Y = \varphi$.

(12) Show that if $G : \Omega \to \mathbb{R}^n$ is C^1, where $\Omega \subset \mathbb{R}^n$ is open, and $G^{-1} : G(\Omega) \to \Omega$ exists and is C^1, then for any rectangle $A \subset \Omega$ that

$$m(G(A)) \leq \int_A |\det(G'(x))| \, dm(x),$$

where here m denotes Lebesgue measure on $\mathbb{R}^n$. You may assume without proof the result in the special case that G is a linear operator or a translation.

(13) Define the Borel measure σ on $S^{n-1} = \{x \in \mathbb{R}^n \,\big|\, |x| = 1\}$ that satisfies

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{r=0}^{\infty} \int_{\theta \in S^{n-1}} r^{n-1} f(r\theta) \, d\sigma(\theta) \, dr.$$

State and prove the formula giving the total measure of S^{n-1}.