Algebra Qualifying Examination
August 2020

You are allowed to rely on a previous part of a multi-part problem even if you do not prove the previous part. There are 100 points total.

Groups

1. Let G be a group. The center of G is $C(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$.

a) (3 points) Show that $C(G)$ is a subgroup of G.
b) (3 points) Show that $C(G)$ is a normal subgroup of G.
c) (6 points) Show that if G is a finite group such that $G/C(G)$ is cyclic then G is abelian.

2. (13 points) How many distinct groups of order 35 are there up to isomorphism? Prove your answer.

Rings

3. (12 points) Let R be a commutative ring (with identity). Suppose that $P_1, \ldots, P_n$ are prime ideals in R and let I be an ideal contained in $\cup_{i=1}^{n} P_i$. Show that $I \subset P_i$ for some i.

4. Suppose that F is a field.

a) (4 points) Suppose that F is finite. Show that there exists a nonzero polynomial $f(x) \in F[x]$ such that $f(a) = 0$ for all $a \in F$.
b) (9 points) Suppose that F is infinite. Suppose that $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ is a nonzero polynomial in the indeterminates $x_1, \ldots, x_n$. Show that there exists $a_1, \ldots, a_n \in F$ such that $f(a_1, \ldots, a_n) \neq 0$.

Modules and Linear Algebra

5. (12 points) Suppose that R is a commutative ring (with identity) and

$$0 \to M' \stackrel{\alpha}{\to} M \stackrel{\beta}{\to} M'' \to 0$$

is a short exact sequence of R-modules and M' and M'' are finitely generated R-modules. Show that M is a finitely generated R-module.

6. (13 points) Let F be a field and A be a nonzero $l \times m$ matrix with coefficients in F. Suppose that $1 \leq n \leq \max\{l, m\}$. An $n \times n$ submatrix of A is a matrix obtained by removing $l - n$ rows and $m - n$ columns from A. The rank of A is the common dimension of the row space and column space of A. Show that

$$\text{rank } (A) = \max\{n \mid \text{Det}(B) \neq 0 \text{ for some } n \times n \text{ submatrix } B \text{ of } A\}.$$

As an example, in the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

the matrix

$$\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$

is a 2×2 *submatrix and* (5) *is a* 1×1 *submatrix.*

Fields

7. (12 points) Suppose that F is a field of characteristic zero and K is a finite field extension of F. Show that there are only finitely many intermediate fields L between F and K.

8. Let K be a splitting field of $f(x) = x^4 - 2x^2 + 9$ over $\mathbb{Q}$. You may assume the fact that $f(x)$ is irreducible in $\mathbb{Q}[x]$ (you do not need to prove that $f(x)$ is irreducible).

a) (3 points) Compute the index $[K : \mathbb{Q}]$.
b) (2 points) Compute the order of the Galois group $\text{Gal}(K/\mathbb{Q})$.
c) (8 points) Compute the group $\text{Gal}(K/\mathbb{Q})$. If it is isomorphic to a well known group, identify the group.