May 2017 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 5 questions with 4 parts. Each part is worth 5 points for a total of 100 points. There are three pages including this one.

In what follows all rings R are assumed to have a multiplicative identity 1_R. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example.

1. Groups.

a. Prove that if G is a simple group and $H < G$ is a proper subgroup of index n, then G is isomorphic to a subgroup of S_n.

b. Prove that a group of order 48 is not simple.

c. Prove or give a counterexample. A group with a finite number of subgroups is finite.

d. Prove or give a counterexample. The product of two elements of finite order in a group always has finite order.

2. Rings.

a. Prove or give a counter-example. If $S \subset R$ is a multiplicatively closed set, then the natural map $R \to S^{-1}R$ is injective.

b. Prove or give a counterexample. If R is an integral domain and $f \in R$ is irreducible, then the principal ideal (f) is prime.

c. Prove that if R is an integral domain and $f \in R[x]$ has degree n, then f has at most n roots in R.

d. Give an example of a ring R and a non-zero polynomial $f \in R[x]$ of degree n such that f has more than n roots.

3. Fields.

a. Let $F \supset K$ be a field extension. Define what it means for an element $\alpha \in F$ to be *algebraic* over K.

b. Prove that if α and β in F are algebraic over K, then so are $\alpha + \beta$ and $\alpha\beta$.

c. Define what it means for a finite field extension $F \supset K$ to be Galois. (Any of a number of equivalent definitions is acceptable.)

d. Prove that if a finite extension $F \supset K$ satisfies $F^{\mathrm{Aut}_K F} = K$, then if $\alpha \in F$ has minimal polynomial $f \in K[x]$ then f splits into linear factors in $F[x]$.

4. Modules.

Let R be a commutative ring.

a. Define what it means for an R-module to be *projective*.

b. Prove that if $0 \to A \to B \to P \to 0$ is a short exact sequence of R modules and P is projective, then $B \simeq A \oplus P$.

c. Give an example of two non-zero R-modules, M, N such that $M \otimes_R N = 0$.

d. Prove that a free R-module is flat.

5. Linear Algebra

a. Let

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

be the matrix representing a linear transformation $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis. Write down the matrix representing $\wedge^2\phi$ with respect to the ordered basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.

b. Find the possible Jordan canonical forms for a matrix whose characteristic polynomial is $(x-1)(x^3-1)$.

c. Prove that if A, B are $n \times n$ complex matrices such that $AB = BA$ then A and B have a common eigenvector.

d. Let V be an inner product space. Prove that any orthogonal collection of vectors is linearly independent.