May 2017 Qualifying Examination

If you have any difficulty with the wording of the following problems please contact the supervisor immediately. All persons responsible for these problems, in principle, will be accessible during the entire duration of the exam. Read the whole test. The problems are not in any particular order of difficulty. There are 5 questions with 4 parts. Each part is worth 5 points for a total of 100 points. There are three pages including this one.

In what follows all rings R are assumed to have a multiplicative identity 1_R. When you are doing a problem with multiple parts, you can use an earlier part in the proof of later ones even if you do not prove the earlier part. If you provide a counterexample, you must provide a reasonable explanation as to why your example is in fact a counter example.

1. Groups.

a. Prove or give a counterexample. If every proper subgroup of a group is abelian then the group is abelian.

b. Let $H < G$ be a maximal proper subgroup of a group G. Prove that if H is normal then$[G : H] = p$ for some prime p. Do not assume that G is a finite group.

c. Prove that if $|G| = p^n q$ with p, q prime and $p > q$ then G has a unique normal subgroup of index q.

d. Prove that $|G| = p^n q$ with p, q prime and $p > q$ then G is solvable.

2. Rings.

a. Prove that a finite domain is a field.

b. Let D be a unique factorization domain (UFD). Define what it means for a polynomial $f \in D[x]$ to be primitive.

c. Prove that if D is a UFD and $f, g \in D[x]$ are primitive then fg is also primitive.

d. Prove that if $\mathfrak{p}$ is a prime ideal in a commutative ring R then $R_{\mathfrak{p}}$ is a local ring. (Recall that a ring is local if it has a unique maximal ideal.)

3. Fields.

a. Prove that if G is a finite subgroup of the multiplicative group of units in a field F then G is cyclic.

b. What is the degree of the extension $\mathbb{Q}(i, e^{2\pi i/3})$ over $\mathbb{Q}$? Find and an element $\alpha \in \mathbb{Q}(i, e^{2\pi i/3})$ which generates this extension.

c. Define what it means for a finite field extension $F \supset K$ to be Galois. (Any of a number of equivalent definitions is acceptable.)

d. Prove or give a counter-example. If $F \supset E \supset K$ is a sequence of field extensions and F is Galois over K then F is Galois over E.

4. Modules.

Let R be a commutative ring.

a. Define what it means for an R-module to be *injective*.

b. Prove or give a counter-example. Every R-module is the quotient of an injective module.

c. Prove that if R is a commutative ring and I, J are ideals then $R/I \otimes_R R/J$ and $R/(I+J)$ are isomorphic as R-modules.

d. Define what it means for an R-module to be flat.

5. Linear Algebra

a. Prove or give a counterexample. If A and B are diagonalizable matrices then AB is diagonalizable.

b. Consider the quadratic form on $\mathbb{R}^3$ given by $q(x, y, z) = x^2 + xy + xz + yz$. Write down the matrix of the associated bilinear form with respect to the standard basis of $\mathbb{R}^3$.

c. What are the possible invariant factors and rational canonical forms for a linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^4$ whose characteristic polynomial is $x^4 - 1$?

d. Prove that if $v_1, \ldots, v_n$ is an orthonormal basis for an n-dimensional vector space V then for any $x \in V$

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

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