Analysis Qualifying Exam - May 2021

Work through all parts included in the 8 items below. Start each item on a new sheet of paper. Your work will be graded for correctness, completeness, and clarity.

1. a) Define the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$. b) Let $f : \mathbb{R} \to \mathbb{R}$, and let the codomain of f be equipped with $\mathcal{B}_{\mathbb{R}}$. Define $\sigma(f)$, the pull-back σ-algebra generated by f. c) Let f be as in b) and also continuous and increasing (i.e. if $x \leq y$ then $f(x) \leq f(y)$). Show that $\sigma(f) = \mathcal{B}_{\mathbb{R}}$ if and only if f is *strictly* increasing.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. a) Explain what it means for a property to hold "almost everywhere". b) Explain what it means for $(X, \mathcal{M}, \mu)$ to be complete. c) Let $(X, \mathcal{M}, \mu)$ be complete. Show that if $E \cup N \in \mathcal{M}$ and $\mu(N) = 0$ then $E \in \mathcal{M}$. Make sure you explain how you use the completeness hypothesis.

3. a) Define "measurable function" from one measurable space to another. b) Define Lebesgue measurability of a function $f : \mathbb{R} \to \mathbb{R}$. c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable then the set $A = \{x \in \mathbb{R} : \text{there is } n \in \mathbb{N} \text{ such that } |f(x)| = n\}$ is Lebesgue measurable.

4. On a measure space $(X, \mathcal{M}, \mu)$ recall that L^+ denotes the class of measurable functions $f : X \to [0, +\infty]$. a) Define $\displaystyle\int_X f d\mu$ and $\displaystyle\int_E f d\mu$, with $E \in \mathcal{M}$, first for f simple in L^+, and then for any $f \in L^+$. b) Show that if $E \in \mathcal{M}$ and $\mu(E) = 0$ then $\displaystyle\int_E f d\mu = 0$, first for f simple in L^+, and then for any $f \in L^+$.

5. a) Define the average operator and the Hardy-Littlewood Maximal function. b) Prove the Maximal Theorem (weak type estimate for the maximal function).

6. a) State the Lebesgue-Radon-Nikodym Theorem for signed measures. b) Define the Radon-Nikodym derivative. c) Let $\{r_n\}_1^\infty$ be any enumeration of $\mathbb{Q}$, the set of rational numbers. Explain why the Lebesgue measure m and the measure $\nu = \sum_1^\infty \delta_{r_n}$ (as defined on the Lebesgue $\sigma-$algebra $\mathcal{L}$) are mutually singular. (Here δ_a denotes the Dirac measure at a.)

7. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed vector spaces. a) Define what it means for a linear operator $T : X \to Y$ to be bounded and define $\|T\|$. b) Consider the operator $T : L^1([0,1]) \to L^\infty([0,1])$ (each space endowed with its natural norm) defined as

$$Tf(x) = \int_0^x \frac{2+t}{1+t} f(t)dt, \qquad x \in [0,1].$$

   Show that T is continuous and $\|T\| = 2$.

8. a) State the Riesz Representation Theorem on a Hilbert space H, and explain how it is used to identify H with its dual H^*. b) Prove the Riesz Representation Theorem.